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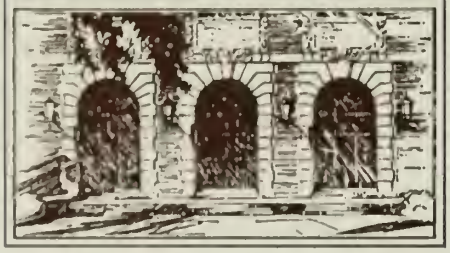

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(This report is being submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics, June 1960.)

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REPORT NO. 100

SINGULAR SHOCK INTERSECTIONS IN PLANE FLOW

by

C. William Gear

May 17, 1960

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I INTRODUCTION

1.1 The purpose of the work reported in the following is to examine the import of various assumptions on the nature of two-dimensional stationary hydrodynamic flow. We assume throughout that the fluid under consideration is a continuum, that its behavior is governed by the equations of conservation of mass, momentum and energy, and that the second law of thermodynamics is applicable. We further assume that the flow takes place in a plane π; that is, the flow variables are independent of distance perpendicular to π and all quantities are functions of two coordinates. We will consider a section of three space parallel to π which has unit thickness perpendicular to π. In the material given below a unit cross section of the fluid will refer to an element of volume in this section.

The conservation equations will be stated in a Cartesian coordinate system. Let Γ be a closed Jordan curve fixed with respect to the set of coordinates x_1 and x_2. Specifically the conservation equations state that the total rate of mass, momentum or energy entering the region bounded by Γ is zero, the force acting on the boundary or the work being done on the boundary respectively. The rate of transfer of mass across a unit straight line whose unit normal has direction cosines n_i is $\rho u_i n_i$ so that the conservation of mass is

$$\oint_{\Gamma} \rho u_i n_i ds = 0 \qquad (1.1.1)$$

where ρ is the mass per unit volume of an $x_1 - x_2$ unit cross section of the fluid, u_i are the components of the velocity referred to the Cartesian coordinates, and the summation convention is used for Roman suffices running through 1 and 2.

The ith component of momentum flow across a unit line is $\rho u_i u_j n_j$ and the ith component of the force acting on a unit line is $\tau_{ij} n_j$ where

$$\tau_{ij} = \delta_{ij}(p - \frac{2}{3} w u_{k,k}) + w(u_{i,j} + u_{j,i})$$

so that the conservation of momentum is

$$\oint_\Gamma (\rho u_i u_j + \tau_{ij}) n_j ds = 0 \qquad (1.1.2)$$

where w is the viscosity of the gas,

p the pressure

and $Y_{,j}$ means $\frac{\partial Y}{\partial x_j}$

The rate of flow of energy across the unit line is $\rho n_i u_i v^2/2$ due to the kinetic energy and $p u_i n_i/(\gamma - 1)$ due to the internal energy, so that the conservation of energy can be written as

$$\oint_\Gamma [(\rho \frac{v^2}{2} + \frac{p}{\gamma - 1}) \delta_{ij} + \tau_{ij}] n_j u_i ds = 0 \qquad (1.1.3)$$

where γ is the ratio of specific heats of the gas,

v is the velocity of the gas

and the heat conductivity is assumed zero.

The second law of thermodynamics implies that the entropy $p\rho^{-\gamma}$ will never decrease in an area moving with the flow. Throughout we will assume that the flow variables ρ, u_i and p are bounded.

1.2 First we will make the additional assumptions that the viscosity w is zero and that the velocity and pressure are single-valued except on a finite number of piecewise analytic curves. If AB is an analytic segment of one of these curves with direction cosines λ_i, and if PQ and SR are representations of AB displaced a distance d along the normal to AB at each point, to the

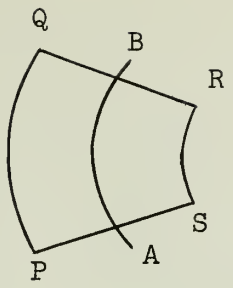

Figure 1.1

left for PQ and the right for SR such that PQRS encloses no other discontinuities, then we can write the conservation equations in the form

$$-\oint_{PQRS} X\,ds = \int_{SR} X\,ds + \int_{QP} X\,ds + \int_{RQ} X\,ds + \int_{PS} X\,ds = 0$$

where X can be the mass flow, momentum flow plus force, or the energy flow plus work. Since this is bounded we have as d tends to zero $\int_{SR} X\,ds = \int_{PQ} X\,ds$, or writing X^- for the limit on the left of AB and X^+ for the limit on the right we get

$$\int_A^B (X^+ - X^-)\,ds = 0 .$$

Since this is true for any points A and B such that AB is analytic, we have $X^- = X^+$ almost everywhere. These are the Rankine-Hugoniot equations, or R-H equations, and imply that either the discontinuity is a stream line of the flow and $\overset{+}{-}[p] = 0$ almost everywhere where $\overset{+}{-}[p] = p^+ - p^-$, or that the discontinuity is a shock described by the equations

$$U_t = U_t^- \tag{1.2.1}$$

$$\frac{p}{p^-} = y = 1 + \frac{2\gamma}{\gamma+1}(\sigma^2 - 1) \tag{1.2.2}$$

$$\frac{\rho}{\rho^-} = \eta = \frac{(\gamma+1)\sigma^2}{2 + (\gamma-1)\sigma^2} \tag{1.2.3}$$

$$\frac{U_n}{\sigma} = \frac{c^-}{\eta} \qquad (1.2.4)$$

almost everywhere, where $c^2 = \frac{\gamma p}{\rho}$, $\sigma = \frac{U_n^-}{c^-}$, U_t and U_n are the components of the velocity tangential and normal to the shock respectively, and the superfix $^+$ has been dropped. The second law of thermodynamics implies that $y \geq 1$, or equivalently that $\sigma \geq 1$.

In regions where the flow variables are differentiable we will transform the independent variables x_1 and x_2 to s and T where T is defined as 0 along an arbitrary curve which is nowhere tangential to a stream line and meets each stream line once and only once. The T coordinate of any point is the arc length from $T = 0$ along the unique stream line to that point so that the transformation is given by $\frac{\partial x_i}{\partial T} = \frac{u_i}{v}$ and the value of $x_1(s, 0)$ and $x_2(s,0)$ as functions of s. Thus the s coordinate is defined for a stream line and hence for each point on it by prescribing s along the curve $T = 0$. We define

$$\lambda_i = \frac{\partial x_i}{\partial s}$$

$$\lambda^2 = (\xi_5)^2 = \lambda_i \lambda_i$$

$$\mu = \xi_4 \text{ by}$$

$$\sin \mu = \frac{\lambda_i u_i}{\lambda v}$$

$$\cos \mu = \lambda_j \frac{u_i \epsilon_{ij}}{\lambda v}$$

$$\xi_1 = \frac{1}{\gamma} \log p$$

$\xi_2 - \frac{\pi}{2}$ to be the angle between a stream line and the x_1 axis

$m = (\xi_3)^{1/2} = \frac{v}{c}$ to be the local Mach number

and $c = \frac{\xi_6}{m}$ is the local velocity of sound.

Making these substitutions into the conservation equations (1.1.1) - (1.1.3) Taub[11] has shown them to become

$$\frac{\partial \zeta_i}{\partial s} = a_{ij} \frac{\partial \xi_j}{\partial s} \qquad i = 1, 2 \tag{1.2.5}$$

where

$$a_{ij} = \lambda \begin{bmatrix} \sin\mu & -m^2\cos\mu \\ (\frac{1}{m^2} - 1)\cos\mu & \sin\mu \end{bmatrix} \tag{1.2.6}$$

$$\frac{\partial m}{\partial T} = -\frac{1}{m}\left(1 + \frac{\gamma - 1}{2} m^2\right) \frac{\partial \xi_1}{\partial T} \tag{1.2.7}$$

$$\frac{\partial c}{\partial T} = \frac{\gamma - 1}{2} c \frac{\partial \xi_1}{\partial T} \tag{1.2.8}$$

plus the two geometrical relationships

$$\frac{\partial \lambda}{\partial T} = \cos\mu \frac{\partial \xi_2}{\partial s} \tag{1.2.9}$$

and

$$\frac{\partial \mu}{\partial T} = \frac{\partial \xi_2}{\partial T} - \frac{\sin\mu}{\lambda} \frac{\partial \xi_2}{\partial s} \tag{1.2.10}$$

When $\det||a_{ij}|| = \lambda^2(1 - m^2\cos^2\mu)$ is non-zero (as it certainly is when $m < 1$) $||a_{ij}||$ can be inverted to give

$$\frac{\partial \xi_\nu}{\partial T} = A_{\nu j} \frac{\partial \xi_j}{\partial s} \qquad \nu = 1, 2, \ldots, 6, \tag{1.2.11}$$

$$A_{\nu j} = \frac{1}{\lambda(1-m^2\cos^2\mu)} \begin{bmatrix} \sin\mu & m^2\cos\mu \\ 1 - \frac{1}{m^2}\cos\mu & \sin\mu \\ -[2 + (\gamma-1)m^2]\sin\mu & m^2[2 + (\gamma-1)m^2]\cos\mu \\ (1 - \frac{1}{m^2})\cos\mu & -m^2\sin\mu\cos^2\mu \\ 0 & \cos\mu \\ -\frac{c}{m}\sin\mu & -mc\cos\mu \end{bmatrix}$$

There exists a class of time dependent flows in two dimensions where the time t enters only in the variables $a_1 = x_1/t$ and $a_2 = x_2/t$. This is called pseudo-stationary flow. If a pseudo-stationary velocity vector U_i is defined as $u_i - a_i$, then if U_i and a_i are used as variables instead of u_i and x_i the equations of flow have been shown by Taub [11] to be of the above form with additional terms not involving derivatives, so that they can be written as

$$\frac{\partial \xi_\nu}{\partial T} = A_{\nu j} \frac{\partial \xi_j}{\partial s} + H_\nu$$

where H_ν is a function of the flow variables. At the end of Chapter II the analogy between stationary and pseudo-stationary flow will be discussed.

1.3 These stationary and pseudo-stationary equations are elliptic or hyperbolic as the Mach number m is smaller or larger than 1. For a Mach number of exactly 1 they are parabolic.

The boundaries of the flow are combinations of shocks and walls which do not in general completely surround the region being considered. However the problem can still be well defined if the subsonic region can be bounded by a combination of shocks, walls and the sonic line. The supersonic region does not need a complete boundary since the equations are hyperbolic there so that an initial value problem can be solved.

Typical configurations that have been observed in shock tubes and wind tunnels are the regular and Mach reflection patterns and the shock attached to or detached from a wedge. In reflection an incident shock I is reflected at a wall W in one of two ways. For some incident shock angles a second shock R can be found that will turn the flow between I and R parallel to the wall. (Figure 1.2) For other incident angles the pattern of Figure 1.3 is observed. Here the junction T of the incident and reflected shock has moved away from

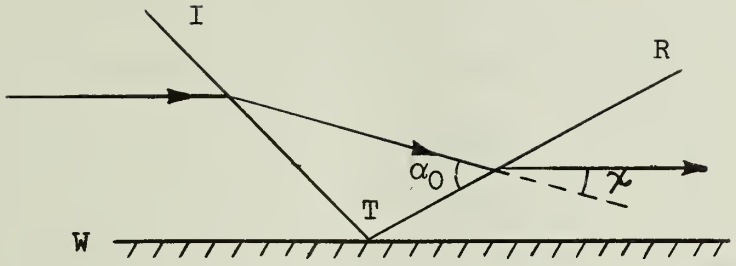

Figure 1.2

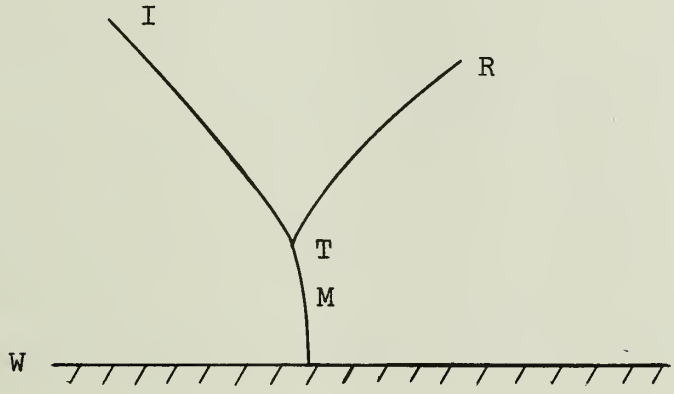

Figure 1.3

the wall and is connected to the wall by the Mach shock M. This is the Mach reflection pattern.

The problem of a wedge in a uniform flow is similar in may ways. For some oncoming flows and wedge angles attached shocks AB are possible. (See Figure 1.4) This corresponds to the existence of the regular reflection of

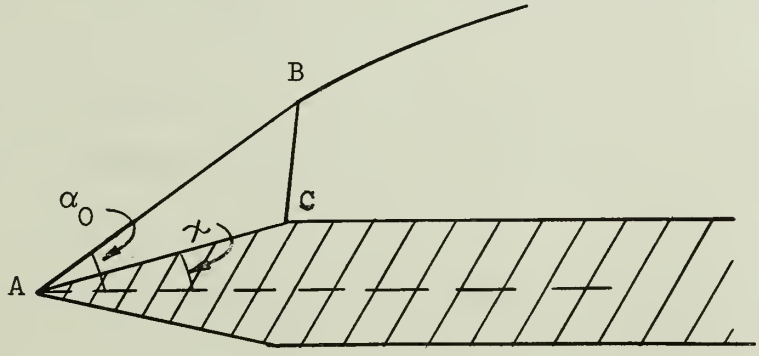

Figure 1.4

Figure 1.2. The oncoming flow in the wedge problem corresponds to the flow behind the incident shock in the regular reflection. The function of the attached shock or the reflected shock is to turn the flow through an angle so that it is parallel to the wall or the wedge. If we think of the wall as being the side AC of the wedge, then the problems are identical.

When an attached shock is no longer possible a detached shock can be observed in front of the wedge that must result in the velocity being zero at the apex of the wedge. This corresponds loosely to a three shock configuration, the Mach reflection phenomenum, where the triple point T detaches from the wall. (See Figure 1.3). In the three shock configuration there is always a trivial configuration where the reflected shock R is a Mach wave and the Mach shock M is a continuation of the incident shock I. If the incident shock is sufficiently weak this is the only theoretical solution, but experimentally other solutions have been observed. We shall return to this point in subsequent discussions.

In Chapter II we will use the equations of Section 1.2 to study the behavior of non-viscous flow around a wedge in a supersonic oncoming flow. This is equivalent to the problem of regular reflection. Buseman[2] has surveyed the results that have been obtained in the wedge problem. When the wedge angle is small there are two shocks possible, one weak and one strong. He points out that both have been obtained experimentally. As the wedge angle χ increases the angles α_0 of these shocks approach each other until they are equal at the so-called extreme angle. For larger χ no attached shock is possible.

If χ is small enough the flow behind the weak shock is supersonic and the shock is straight from the apex A of the wedge (see Figure 1.4) to the point B where the characteristic leaving the shoulder C of the wedge meets

the shock. From there it consists of a number of analytic segments at least until the flow becomes sonic. In Section 2.1 this will be shown in some detail. As χ increases the angle of the weak shock α_0 increases until the flow behind the shock is sonic at $\alpha_0 = \alpha_s$ the sonic angle. If χ is increased still further the shock is curved all of the way down to the point A. Thomas[12] and Taub[11] point out that only for a countable number of angles χ is it possible to have an analytic shock that is curved and yet have a straight stream line AC. Guderly[6] [7] has studied the singular flow in ABC when the oncoming flow is nearly sonic and has obtained a solution such that the curvature of the shock at A is zero for α_0 between the sonic angle and the Crocco[4] angle. At the Crocco angle the shock may have a finite curvature and the solution be regular. For α_0 greater than this but less than the extreme angle which is the α_0 corresponding to the largest possible deflection χ of the flow, the shock may have an infinite curvature at A.

Later in Chapter II we discard the assumption that the shock is piecewise analytic when it bounds a subsonic region, and introduce a non-analytic transformation of the s-T plane. In this new plane we assume that we have a singularity that has the nature of a branch point. The effect of this assumption is that in the physical plane the flow variables are assumed single-valued behind the shock although their derivatives have values at the singularity that depend on the direction in which it is approached. The angle of the shock thus has a limit as one approaches the apex A of the wedge and this angle must be such that the flow emerging from the shock is parallel to the wedge wall at A. This does not allow the type of singularity where the pressure or direction of flow is multiple-valued behind the shock to exist. Consequently the angle of the shock at the apex of the wedge is determined by the R-H equations and the fact that the flow leaving the shock must be parallel to the wall.

Singularities of the type treated in Chapter II cannot be used in discussing three shock configurations occurring in Mach reflection of weak incident shocks since one would have to postulate the existence of a non-trivial three shock configuration in order to place such a singularity at the triple point of the configuration of shocks in this case.

The local boundary conditions, that is the flow ahead of the shock and the shape of the wall in the neighborhood of the apex of the wedge, determine a class of solutions in the new plane. This class includes the regular solutions and others. The solutions to some of the boundary value problems may require shocks having an infinite curvature.

1.4 One problem that this method will fail to provide a solution for is that of a curved wedge in a uniform oncoming flow when the angle of the wedge satisfies a certain condition. There are a countable number of such angles and unless additional restrictions are satisfied by the flow and wedge profile there are no solutions of the class discussed. Rejecting the idea that nature will discriminate against this set of wedge angles we must conclude either that there are solutions with other forms of singularities or that the mathematical model of the fluid is inadequate.

One type of singularity that we might look for is one where the angle of the flow and other flow variables are not single-valued everywhere behind the shock. Such a flow, the Prandtl-Meyer[1] flow, exists in the supersonic region, but we are concerned with the subsonic region where no such flow is known at present. We can try and find such singularities and try and decide if the mathematical model is adequate jointly by examining viscous flow and in particular studying the solution as the viscosity tends to zero.

There is a reason to believe that the solution of a problem for a finite viscosity will not approach the solution of the equivalent non-viscous

problem as the viscosity tends to zero. This is that the curvature of the shock in some of the solutions discussed can be infinite. If the flow is one-dimensional, that is if all derivatives tangential to the shock are zero, then the R-H equations are nearly satisfied across a shock region, where by a shock region we mean the region between two bounding lines $T = T_0$ and $T = T_1$ which encompass most of the change in the flow. $X^+ - X^- = f(T_1 - T_0)$ where $f(T_1 - T_0)$ tends to zero as $T_1 - T_0$ tends to infinity. If we assign a value to $f(T_1 - T_0)$ then $T_1 - T_0$, the shock thickness, is proportional to the viscosity w. For small shock curvatures k terms will be present that are proportional to the curvature. The region over which the terms act is proportional to w so the effect is proportional to kw. Thus if the viscosity tends to zero the behavior the solution will depend partly on the behavior of kw, and if k becomes infinite when the viscosity is zero, the terms in kw may still play a part.

Sternberg [10] considered the effects of modifying the R-H equations in the neighborhood of the intersection region by introducing viscosity. Using approximate solutions obtained from an electric tank he showed that a large modification was necessary to have an appreciable effect on the flow.

In Chapter III the equations of viscous flow will be discussed. To simplify the problem we will assume that in some regions viscosity can be neglected, and that these regions can be joined together by transition regions in which viscous forces play a role. We will start by examining the transition region analogous to a shock, and then examine ways in which two or three of these regions can interact.

The treatment will differ from Sternberg's [10] treatment in that the Mach and reflected shocks will be allowed to be curved rather than being considered as one-dimensional shocks. The intersection region where all of these shocks interact will correspond to his non-R-H region. He applied the four

conservation equations around this region to evaluate its size and shape using experimental values for the angles of the three shocks, whereas this treatment will attempt to determine the angles of the shocks.

Sternberg discussed the possible effects of this region on the flow downstream and in his specific calculation showed that the region is of considerable size compared to the thicknesses of the shocks. He further argued that the limit of the solution as the viscosity tends to zero may not be the solution for zero viscosity. We too will find that this can be the case for regular and Mach reflection.

Since the arguments presented are local in character, no account is taken of distant boundary conditions. The lengths involved are proportional to viscosity and the shock curvatures are inversely proportional to viscosity. By using polynomial approximations in s and T for the flow variables we will study the problem in detail and exhibit the equations governing two and three shock configurations for a linear polynomial approximation to each of the flow variables p, ξ_2, v, λ and α. The unknowns, which are now the coefficients of the polynomials, will be expressed as power series in the strength of the incident shock so that the equations for regular and Mach reflection can be expressed as algebraic equations in the coefficients of the power series.

For weak shocks we can take the first terms of these power series to get relations involving eight unknowns. Four of these will be variables specifying the angles of the various shock boundaries. The solution for these is independent of the viscosity w. The other four variables are two shock curvatures and two distances. If we write $K = kw$ and $d = D/w$ where D is a typical length we will find that the solutions for K and d are independent of viscosity so that if $K \neq 0$, the limit of the solutions as viscosity tends to zero will be different from the solution of the non-viscous shock intersection problem.

II NON-VISCOUS FLOW

2.1 In this section the solution of the stationary hydrodynamic equations are discussed for a wedge in a uniform oncoming flow which is such that the flow behind the shock is initially supersonic. If the wedge has infinite straight sides then a solution of this type is known to exist, namely the straight weak shock that turns the flow through the angle of the wedge. If now a convex angle appears in the boundary at a point C, such that the wall is composed of two straight portions, the shock will remain straight until a characteristic leaving the point C reaches it at B.

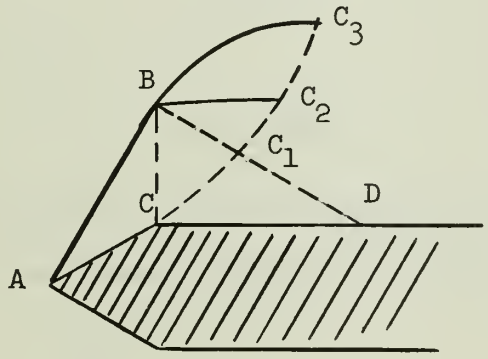

Figure 2.1

The flow in the region ABC is uniform. Let BC_1D be another characteristic leaving the point B. Then the flow in the region CBC_1D can only be affected by the oncoming parallel flow from ABC and the wall between C and D. In the case under consideration a solution is known for BCD, namely a suitable Prandtl-Meyer[1] region BC_1C and a uniform region CC_1D. Hence all derivatives of the flow variables taken along the characteristic BD can be computed. Now it will be shown that the equations of motion and the R-H equations have a solution that is separately analytic in the regions C_3BC_2 and C_2BC_1, where BC_2 is the stream line through the point B.

The coordinates s and T are chosen in the region C_1BC_2 so that T = 0 is the line BC_1. In the physical plane the characteristics subtend an angle M with the stream line, where $M = \sin^{-1}(1/m)$. Thus on BC_1 $\mu = M - \pi/2$ where μ is the angle between the normal to T equal constant (the curve BC_1) and the tangent to s equal constant (the stream line). Since derivatives are known along BC_1 the $\frac{\partial \xi_i}{\partial s}$ are known there. By (1.2.5) we have

$$\frac{\partial \xi_i}{\partial s} = a_{ij} \frac{\partial \xi_j}{\partial T} \tag{2.1.1}$$

where

$$a_{ij} = \lambda \begin{bmatrix} -\cos M & -\frac{1}{\sin M} \\ -\cos^2 M \sin M & -\cos M \end{bmatrix} \tag{2.1.2}$$

as follows from (1.2.6) on substituting for μ and m. Det $||a_{ij}|| = 0$, therefore $\frac{\partial \xi_1}{\partial s}$ and $\frac{\partial \xi_2}{\partial s}$ evaluated along BC_1 will satisfy the relation

$$\frac{\frac{\partial \xi_1}{\partial s}}{\frac{\partial \xi_2}{\partial s}} = \frac{a_{11}}{a_{21}} = \frac{a_{12}}{a_{22}} = \frac{1}{\sin M \cos M} .$$

Thus when we have evaluated either $\frac{\partial \xi_1}{\partial s}$ or $\frac{\partial \xi_2}{\partial s}$ from the Prandtl-Meyer region BCC_1 we can solve (2.1.1) for $\frac{\partial \xi_i}{\partial T}$. Since $||a_{ij}||$ is of rank 1 there will be a one parameter family of solutions. Therefore we can write

$$\frac{\partial \xi_j}{\partial T} = b_j - \beta_1 c_j \tag{2.1.3}$$

where

$$b_j = \begin{bmatrix} -\frac{\partial \xi_1}{\partial s} & \frac{1}{\lambda \cos M} \\ 0 & \end{bmatrix} \quad \text{and} \quad c_j = \begin{bmatrix} -\frac{1}{\cos M} \\ \sin M \end{bmatrix}$$

for any constant β_1. Since the region adjacent to BC_1 is not in general uniform, $\frac{\partial \xi_1}{\partial s} \neq 0$. Between the two regions of analytic flow separated by the stream line $s = 0$ the flow variables ξ_1 and ξ_2 must be such that $\xi_1(T,0) \equiv \xi_2(T,0)$ for all points on the stream line. In particular $\frac{\partial^n \xi_i}{\partial T^n}$ evaluated in one region must be the same as that evaluated in the other region at B, for all $n = 0, 1, 2, \ldots$. For $n = 0$, this implies that the initial angle of the curved shock starting from B is the same as that of the straight shock. For region C_2BC_3 we define $T = 0$ as the shock BC_3. Following Taub [11] we have by differentiation of the R-H equations:

$$\frac{\partial \xi_i}{\partial s} = k a_i \tag{2.1.4}$$

where k is the curvature of the shock and

$$a_1 = \frac{2m^2 \sin 2\mu}{\gamma + 1} \; ; \quad a_2 = \frac{\sin^2 \mu}{\gamma} \left(\frac{4\gamma}{\gamma + 1} + 1 - \gamma \right) - \cos^2 \mu (\gamma - 1) .$$

Inverting (1.2.5) and substituting from (2.1.4) we get

$$\frac{\partial \xi_i}{\partial T} = k A_{ij} a_j \tag{2.1.5}$$

where

$$A_{ij} = \frac{1}{\lambda(1 - m^2 \cos^2 \mu)} \begin{bmatrix} \sin \mu & m^2 \cos \mu \\ \left(1 - \frac{1}{m^2}\right) \cos \mu & \sin \mu \end{bmatrix}$$

Equating (2.1.3) and (2.1.5) we get

$$kA_{ij}a_j = b_i - \beta_1 c_i \qquad (2.1.6)$$

These are two equations in the two unknowns k and β_1. They have a unique solution if and only if

$$\det \begin{bmatrix} A_{1j}a_j & c_1 \\ A_{2j}a_j & c_2 \end{bmatrix} \neq 0 .$$

Since $0 \leq \mu < \pi/2$ in region BC_2C_3, each entry in A_{ij} is positive. Further a_1 and a_2 are both positive in the supersonic region so that $A_{1j}a_j$ and $A_{2j}a_j$ are both positive. Since c_1 and c_2 have opposite signs (2.1.7) will always be satisfied.

Therefore (2.1.6) can always be solved uniquely to give

$$k = \frac{b_1c_2 - c_1b_2}{A_{1j}a_jc_2 - A_{2j}a_jc_1}$$

$$\beta_1 = \frac{b_1A_{2j}a_j - b_2A_{1j}a_j}{A_{1j}a_jc_2 - A_{2j}a_jc_1} .$$

Now suppose that all the partial derivatives of the flow variables of order less than n are known, together with derivatives of k of order less than n-1 such that the equations connecting the derivatives of k and the flow variables are consistent. By successive differentiation of (1.2.5) and (1.2.6) we find that

$$\frac{\partial^n \xi_i}{\partial s^n} = (a_{ij})^n \frac{\partial^n \xi_j}{\partial T^n} + Q_{ni} \qquad (2.1.8)$$

where the Q_{ni} are polynomials in the known lower order derivatives.

At B in region C_1BC_2 the $\frac{\partial^n \xi_i}{\partial s^n}$ are known, therefore

$$\frac{\partial^n \xi_i}{\partial T^n} = \beta_n c_i - b_{ni} \tag{2.1.9}$$

$$b_{ni} = \begin{bmatrix} \frac{2}{(-2\cos M)^n} \left(\frac{\partial^n \xi_1}{\partial s^n} - Q_{n1} \right) \\ 0 \end{bmatrix} .$$

We get by differentiation of the Rankine-Hugoniot equations along the shock BC_3

$$\frac{\partial^n \xi_i}{\partial s^n} = a_i \frac{\partial^{n-1} k}{\partial s^{n-1}} + P_{ni} \tag{2.1.10}$$

where P_{ni} is a polynomial in the known derivatives of lower order.

In the region BC_2C_3 $||a_{ij}||$ is not a singular so (2.1.8) can be inverted to

$$\begin{aligned} \frac{\partial^n \xi_i}{\partial T^n} &= A^n_{ij} \frac{\partial \xi_j}{\partial s^n} + A^n_{ij} Q_{nj} \\ &= A^n_{ij} a_j \frac{\partial^{n-1} k}{\partial s^{n-1}} + R_{ni} . \end{aligned} \tag{2.1.11}$$

Equating (2.1.9) and (2.1.11) we again have two equations in two unknowns $\frac{\partial^{n-1} k}{\partial s^{n-1}}$ and β_n. These have a unique solution if and only if

$$\det \begin{bmatrix} A^n_{ij} a_j & c_1 \\ A^n_{ij} a_j & c_2 \end{bmatrix} \neq 0 . \tag{2.1.12}$$

Following Taub[11] we write

$$\tanh\theta = (m^2 - 1)^{1/2} \cot \mu ,$$

hence

$$A_{ij}^{n} = \frac{1}{\lambda^{n}(1 - m^{2}\cos^{2}\mu)^{n/2}} \begin{bmatrix} \cosh n\theta & \frac{m^{2}}{(1-m^{2})^{1/2}} \sinh n\theta \\ \frac{(m^{2}-1)^{1/2}}{m^{2}} \sinh n\theta & \cosh n\theta \end{bmatrix}$$

In the coordinate system under consideration

$$m^{2}\cos^{2}\mu < 1 , \qquad 0 < \mu < \frac{\pi}{2} ,$$

hence each entry in A_{ij}^{n} is positive. Also

$$a_{1} = \frac{m^{2}\sin^{2}\mu}{\gamma + 1} > 0 ,$$

and a_{2} may be rewritten as

$$\frac{2}{\gamma + 1} \left[(1 - m^{2}\cos^{2}\mu)\left(1 - \frac{1}{m^{2}} + \frac{1}{\sigma^{2}}\right) + (m^{2} - 1)\cos^{2}\mu \left(1 + \frac{1}{\sigma^{2}}\right)\right] > 0$$

when $m^{2} > 1$. Since c_{1} and c_{2} have opposite signs, (2.1.12) cannot vanish in the supersonic region for $n = 1, 2, 3, \ldots$. Thus (2.1.9) and (2.1.11) can always be solved uniquely for $\frac{\partial^{n-1}k}{\partial s^{n-1}}$ and β_{n}, completing one step of the induction process.

The region behind the shock may be systematically covered by regions in which the flow is analytic. These regions start either at the shock (i.e. regions like $BC_{2}C_{3}$) or at a characteristic leaving a discontinuity (i.e. regions like $BC_{1}C_{2}$). They are bounded by a stream line (which may be the wall) and terminated by a characteristic of the other set leaving a discontinuity. By equating conditions across the stream line joining adjacent regions the flow may be determined analytically in these regions. The sides of the wedge can consist of a number of curved analytic segments instead of the straight sides

treated earlier, and until another shock forms, or the characteristics of the same set intersect again (they intersect at C in the Prandtl-Meyer fan), the problems can be treated by the methods given.

2.2 When the flow behind the shock is subsonic we will have

$$\det||a_{ij}|| = \lambda^2(1 - m^2\cos^2\mu) > 0$$

so that $||a_{ij}||$ can be inverted. Multiplying (2.1.8) by $||a_{ij}||^{-n} = A^n_{ij}$ we get

$$\frac{\partial^n \xi_j}{\partial s^n} A^n_{ij} = \frac{\partial^n \xi_i}{\partial T^n} + \bar{P}_{ni}$$

where the $\bar{P}_{ni}$ are polynomials in the known lower order derivatives. Substituting from (2.1.10) into this we get

$$\frac{\partial^n \xi_i}{\partial T^n} = A^n_{ij}a_j \frac{\partial^{n-1} k}{\partial s^{n-1}} + A^n_{ij}P_{nj} - \bar{P}_{ni} \quad . \tag{2.2.1}$$

Thomas[12] and Taub[11] have used this equation and the fact that the $\frac{\partial^n \xi_2}{\partial T^n}$ are given at $s = 0$ for $n \geq 1$ along a wedge wall or known stream line to show that unless $A^n_{2j}a_j \neq 0$ for all $n \geq 1$, no finite $\frac{\partial^{n-1} k}{\partial s^{n-1}}$ may exist to satisfy (2.2.1) for all n.

If the known stream line is straight we have that $\frac{\partial^n \xi_2}{\partial T^n} = 0$ for all $n \geq 1$. If we assume that the first n-1 derivatives of all flow variables and the first n-2 derivatives of k are zero then the P_{nj} and the $\bar{P}_{nj}$ of (2.2.1), are zero, so that (2.2.1) reduces to

$$A^n_{21}a_1 \frac{\partial^{n-1} k}{\partial s^{n-1}} + A^n_{22}a_2 \frac{\partial^{n-1} k}{\partial s^{n-1}} = 0 \quad .$$

Unless $A^n_{2j}a_j = 0$ we must have $\frac{\partial^{n-1} k}{\partial s^{n-1}} = 0$ or ∞. Consequently we can show by induction that unless $A^n_{2j}a_j = 0$ for some n then all derivatives of the curvature

and the curvature itself are zero or infinity. Thomas[12] has shown that the number of wedge angles for which this happens is countable, and that these wedge angles form a dense set.

To study the flow in the neighborhood of the apex A of the wedge we will make a singular transformation of the s-T plane. Let m_1 be the Mach number of the flow behind the shock at A; that is it is the limit of the Mach number of the flow as one approaches A along the shock. Let μ_1 be the value of μ at that point.

Define $$\phi = \frac{\pi}{2} - \mu_1$$

$$R^2 = s^2(1 - m_1^2) \sin^2\phi + (s \cos \phi + T)^2$$

$$\tan \beta = \frac{s(1 - m_1^2)^{1/2} \sin \phi}{s \cos \phi + T}$$

Since the shock cannot be a stream line we know that $\sin \phi \neq 0$ and we are assuming that $m_1^2 < 1$.

If we introduce a further coordinate plane w-z where

$$z = s \sin \phi (1 - m_1^2)^{1/2}$$

$$w = s \cos \phi + T$$

then R and β are polar coordinates in the w-z plane and the mapping of the shock (T = 0) and the wall of the wedge (s = 0) into the w-z plane is given in Figure 2.2. The shock is given by $\beta = \beta_0$ where

$$\tan \beta_0 = (1 - m_1^2)^{1/2} \tan \phi$$

and the wall is given by $\beta = 0$. Since ϕ and m_1 are constants the w-z plane is a linear transformation of the s-T plane.

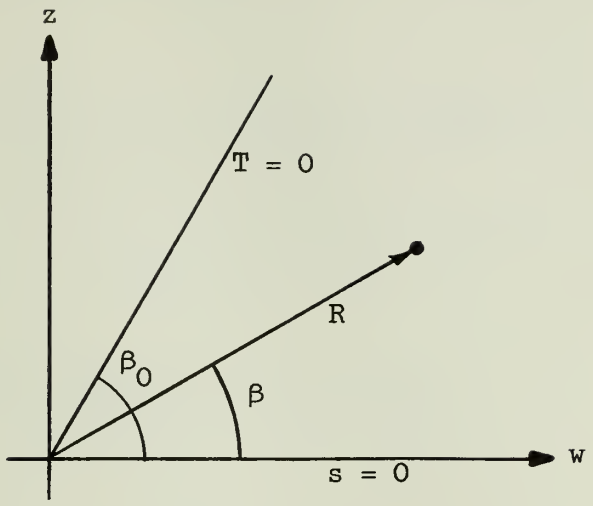

Figure 2.2

We have

$$\frac{\partial R}{\partial s} = \sin \phi (1 - m_1^2)^{1/2} \sin \beta + \cos \phi \cos \beta$$

$$\frac{\partial R}{\partial T} = \cos \beta$$

$$\frac{\partial \beta}{\partial s} = \frac{1}{R} \left\{ \cos \beta \sin \phi (1 - m_1^2)^{1/2} - \cos \phi \sin \beta \right\}$$

and

$$\frac{\partial \beta}{\partial T} = - \frac{1}{R} \sin \beta \ .$$

Thus

$$J\left(\begin{matrix} R, \beta \\ s, T \end{matrix}\right) = - \frac{1}{R} \sin \phi (1 - m_1^2)^{1/2}$$

so the transformation is non-singular except at $R = 0$ and ∞.

We will look for solutions that have a singularity at $R = 0$ which can be written in the form

$$\xi_\nu (R, \beta) = \sum_{n=0}^{\infty} R^{nt} f_{n\nu} (\beta) \qquad \nu = 1, 2, \ldots, 6,$$

where t is any positive constant, $\xi_3 = m^2$, $\xi_4 = \mu$, $\xi_5 = \lambda$ and $\xi_6 = v$.

We can write

$$\frac{\partial \xi_\nu}{\partial s} = \sum_{n=0}^{\infty} R^{nt-1} \Big\{ ntf_{n\nu}(\sin\beta \sin\phi(1 - m_1^2)^{1/2} + \cos\beta\cos\phi) + \dot{f}_{n\nu}(\sin\phi\cos\beta(1 - m_1^2)^{1/2} - \cos\phi\sin\beta) \Big\}$$

$$\frac{\partial \xi_\nu}{\partial T} = \sum_{n=0}^{\infty} R^{nt-1} \Big\{ ntf_{n\nu}\cos\beta - \dot{f}_{n\nu}\sin\beta \Big\}$$

where $\dot{f} = \frac{df}{d\beta}$.

We assume that $f_{03} = (m^2)_{R=0} \neq 0$ and substitute the above relations in equations (1.2.11). We will then write the resulting equations as power series in R and equate each coefficient to 0. On substituting into (1.2.11) we have

$$\sum_{n=0}^{\infty} R^{nt-1} \Big\{ ntf_{n\nu}\cos\beta \div \dot{f}_{n\nu}\sin\beta \Big\}$$

$$= \sum_{n=0}^{\infty} A_{\nu j} R^{nt-1} \Big\{ ntf_{nj}(\sin\beta\sin\phi(1 - m_1^2)^{1/2} + \cos\beta\cos\phi) + \dot{f}_{nj}(\cos\beta\sin\phi(1 - m_1^2)^{1/2} - \sin\beta\cos\phi) \Big\}.$$

Now $A_{\nu j}$ is a function of ξ_3, ξ_4, ξ_5 and ξ_6 and so can be written in the form

$$A_{\nu j} = \sum_{q=0}^{\infty} A_{q\nu j} R^{qt}$$

where $A_{q\nu j}$ is independent of R and is a function of the $f_{0\nu}$, $f_{1\nu}$, ..., $f_{q\nu}$ only. Therefore we can write the flow equations above as

$$\sum_{n=0}^{\infty} R^{nt-1}\, ntf_{n\epsilon}[\delta_{\nu\epsilon}\cos\beta - \sum_{q=0}^{\infty} R^{qt}A_{q\nu j}\delta_{j\epsilon}(\sin\beta\sin\phi(1-m_1^2)^{1/2} + \cos\phi\cos\beta)]$$

$$= \sum_{n=0}^{\infty} R^{nt-1}\dot{f}_n\,[\delta_{\nu\epsilon}\sin\beta + \sum_{q=0}^{\infty} R^{qt}A_{q\nu j}\delta_{j\epsilon}(\sin\phi\cos\beta(1-m_1^2)^{1/2} - \cos\phi\sin\beta)] \qquad (2.2.2)$$

where the summation convention is used for Roman suffices running through 1 and 2 for Greek suffices running from 1 through 6.

If we evaluate the term in R^{-1} we get a set of six total differential equations for the $f_{0\nu}$ which can be written in the form

$$\sum_{\epsilon=1}^{6} \bar{E}_{\nu\epsilon}\dot{f}_{0\epsilon} = 0 \qquad \nu = 1, 2, \ldots, 6.$$

where

$$\bar{E}_{\nu\epsilon} = a_{\nu\tau}[\delta_{\tau\epsilon}\sin\beta + A_{0\tau j}\delta_{j\epsilon}(\sin\phi\cos\beta(1-m_1^2)^{1/2} - \cos\phi\sin\beta)]$$

where $a_{\nu\tau} = \delta_{\nu i}((a_{ij})_{R=0} + \delta_{ij})\delta_{j\tau} - \delta_{\nu\tau}$ and is given by

$\bar{E}_{\nu\epsilon} =$

$$\begin{bmatrix}
\bar{E}_{11} & -f_{05}f_{03}\cos f_{04}\sin\beta & 0 & 0 & 0 & 0 \\
f_{05}(\frac{1}{f_{03}} - 1)\cos f_{04}\sin\beta & \bar{E}_{22} & 0 & 0 & 0 & 0 \\
\bar{E}_{31} & \bar{E}_{32} & -\sin\beta & 0 & 0 & 0 \\
\bar{E}_{41} & \bar{E}_{42} & 0 & -\sin\beta & 0 & 0 \\
\bar{E}_{51} & \bar{E}_{52} & 0 & 0 & -\sin\beta & 0 \\
\bar{E}_{61} & \bar{E}_{62} & 0 & 0 & 0 & -\sin\beta
\end{bmatrix}$$

where $\bar{E}_{11} = \bar{E}_{22} = \cos\beta\sin\phi(1-m_1^2)^{1/2} - \sin\beta(\cos\phi - f_{05}\sin f_{04})$

and the other $\bar{E}_{\nu\epsilon}$'s indicate non-zero terms unimportant to the discussion.

Now

$$\det_{i,j=1,2} ||\bar{E}_{ij}|| = \left\{ \sin\phi \cos\beta(1 - m_1^2)^{1/2} - \sin\beta(\cos\phi - f_{05} \sin f_{04}) \right\}^2$$

$$+ f_{05}^2(1 - f_{03}) \cos^2 f_{04} \sin^2\beta$$

$$> 0 \quad \text{since} \quad f_{03} = m^2{}_{R=0} < 1 \quad \text{by hypothesis.}$$

Since in addition $\bar{E}_{i\nu} = 0$ if $i = 1$ or 2 and $\nu = 3, 4, 5$ or 6, we have $\dot{f}_{01} = \dot{f}_{02} = 0$ for all β, so that f_{01} and f_{02} are constants. Since

$$\det||\bar{E}_{\nu\epsilon}|| = \sin^4\beta \det||\bar{E}_{ij}|| > 0$$

if $\sin\beta \neq 0$, $\dot{f}_{0\nu} = 0$ for all β such that $\sin\beta = 0$.

Confining our attention to a region where $0 < \beta < \pi$ we have at $\beta = \beta_0$ (the shock) $\cos\phi = \sin\mu_1$, $\lambda = f_{05} = 1$, and $f_{04} = \mu_1$ so that $\cos\phi = f_{05}\sin f_{04}$. Making this substitution into (2.2.2) and rearranging we get

$$\sum_{n=0}^{\infty} R^{nt-1} [E_{\nu\epsilon} \dot{f}_{n\epsilon} - ntB_{\nu\epsilon} f_{n\epsilon} - C_{n\nu}] = 0 \qquad (2.2.3)$$

where

$$E_{\nu\epsilon} = a_{\nu\tau}[\delta_{\tau\epsilon} \sin\beta + A_{0\tau j}\delta_{j\epsilon}(\sin\phi(1 - m_1^2)^{1/2} \cos\beta - \cos\phi \sin\beta)]$$

$$B_{\nu\epsilon} = a_{\nu\tau}[\delta_{\tau\epsilon} \cos\beta - A_{0\tau j}\delta_{j\epsilon}(\sin\beta \sin\phi(1 - m_1^2)^{1/2} + \cos\phi \cos\beta)]$$

and

$$C_{n\nu} = -a_{\nu\tau} \sum_{q=1}^{n} A_{q\tau j}\delta_{j\epsilon} \Big\{ \dot{f}_{n-q\epsilon}(\sin\phi \cos\beta(1 - m_1^2)^{1/2} - \cos\phi \sin\beta)$$

$$+ (n - q)tf_{n-q\epsilon}(\sin\beta \sin\phi(1 - m_1^2)^{1/2} + \cos\beta \cos\phi) \Big\}.$$

Thus $C_{n\nu}$ is a polynomial in the $f_{0\epsilon}$, $f_{1\epsilon}$, ..., and $f_{n-1\epsilon}$. The matrices E and B are given below:

$E =$

$$\begin{bmatrix}
(1 - m_1^2)^{1/2}\sin\phi\cos\beta & -m_1^2 \sin\phi\sin\beta & 0 & 0 & 0 & 0 \\
(\frac{1}{m_1^2} - 1)\sin\phi\sin\beta & (1 - m_1^2)^{1/2}\sin\phi\cos\beta & 0 & 0 & 0 & 0 \\
-[2 + (\gamma-1)m_1^2]\sin\beta & 0 & - \sin\beta & 0 & 0 & 0 \\
0 & E_{24} & 0 & - \sin\beta & 0 & 0 \\
0 & E_{25} & 0 & 0 & - \sin\beta & 0 \\
- \frac{c_1}{m_1} \sin\beta & 0 & 0 & 0 & 0 & - \sin\beta
\end{bmatrix}$$

where $E_{24} = \cos\phi \sin\phi \cos\beta(1 - m_1^2)^{1/2} + \sin^2\phi \sin\beta$

and $E_{25} = -\sin^2\phi \cos\beta(1 - m_1^2)^{1/2} + \sin\phi \cos\phi \sin\beta.$

$B =$

$$\begin{bmatrix}
- \sin\beta\sin\phi(1 - m_1^2)^{1/2} & -m_1^2\sin\phi\cos\beta & 0 & 0 & 0 & 0 \\
(\frac{1}{m_1^2} - 1)\sin\phi\cos\beta & -\sin\beta\sin\phi(1 - m_1^2)^{1/2} & 0 & 0 & 0 & 0 \\
-[2 + (\gamma -1)m_1^2]\cos\beta & 0 & - \cos\beta & 0 & 0 & 0 \\
0 & B_{24} & 0 & - \cos\beta & 0 & 0 \\
0 & B_{25} & 0 & 0 & - \cos\beta & 0 \\
- \frac{c_1}{m_1} \cos\beta & 0 & 0 & 0 & 0 & - \cos\beta
\end{bmatrix}$$

where $B_{24} = -\cos\phi\sin\phi\sin\beta(1 - m_1^2)^{1/2} + \sin^2\phi\cos\beta$

and $B_{25} = \sin^2\phi\sin\beta(1 - m_1^2)^{1/2} + \sin\phi\cos\phi\cos\beta.$

In particular $C_{0\nu} = C_{1\nu} = 0$.

Now

$$\det||E_{\nu\epsilon}|| = (1 - m_1^2)\sin^2\phi \sin^4\beta \neq 0$$

in the region considered, so we may invert it to get

$$\dot{f}_{n\nu} = nt\, D_{\nu\epsilon} f_{n\epsilon} + \bar{C}_{n\nu} \qquad (2.2.4)$$

from each term of (2.2.3), where the $\bar{C}_{n\nu} = E^{-1}_{\nu\epsilon} C_{n\epsilon}$ are also polynomials in sin β, cos β, the $f_{m\epsilon}$ and $\dot{f}_{m\epsilon}$ for m = 1, 2, ..., n-1, and the matrix $D = E^{-1}B$ and is given by

$$\begin{bmatrix}
0 & -\dfrac{m_1^2}{(1 - m_1^2)^{1/2}} & 0 & 0 & 0 & 0 \\
\dfrac{(1 - m_1^2)^{1/2}}{m_1^2} & 0 & 0 & 0 & 0 & 0 \\
\cot\beta[2 + (\gamma - 1)m_1^2] & \dfrac{m_1^2[2 + m_1^2(\gamma - 1)]}{(1 - m_1^2)^{1/2}} & \cot\beta & 0 & 0 & 0 \\
D_{14} & D_{24} & 0 & \cot\beta & 0 & 0 \\
D_{15} & D_{25} & 0 & 0 & \cot\beta & 0 \\
\dfrac{c_1}{m_1}\cot\beta & \dfrac{c_1 m_1}{(1 - m_1^2)^{1/2}} & 0 & 0 & 0 & \cot\beta
\end{bmatrix}$$

where $$D_{14} = \frac{(1 - m_1^2)^{1/2}}{m_1^2}[\sin^2\phi + \sin\phi\cos\phi\cot\beta(1 - m_1^2)^{1/2}]$$

$$D_{24} = -[\sin^2\phi\cot\beta - \sin\phi\cos\phi(1 - m_1^2)^{1/2}]$$

$$D_{15} = -\frac{(1 - m_1^2)^{1/2}}{m_1^2}[\sin^2\phi\cot\beta(1 - m_1^2)^{1/2} - \sin\phi\cos\phi]$$

$$D_{25} = -[\sin^2\phi(1 - m_1^2)^{1/2} + \sin\phi\cos\phi\cot\beta] \quad .$$

In particular $\bar{C}_{0\nu} = \bar{C}_{1\nu} = 0$.

The solution of (2.2.4) can be written as a particular solution plus arbitrary multiples of the six independent solutions of the homogeneous equations obtained by setting $\bar{C}_{n\nu} = 0$. Suppose a particular solution is $F_{n\nu}$, then we may write the general solution as

$$f_{n\nu} = F_{n\nu} + \alpha_{n\epsilon}F_{n\nu\epsilon} \qquad (2.2.5)$$

where the $\alpha_{n\epsilon}$ are arbitrary constants and the matrix $F_{n\nu\epsilon}$ is

$$\begin{bmatrix}
m_1^2 \sin nt\beta & m_1^2\cos nt\beta & 0 & 0 & 0 & 0 \\
-(1-m_1^2)^{1/2}\cos nt\beta & (1-m_1^2)^{1/2}\sin nt\beta & 0 & 0 & 0 & 0 \\
F_{n31} & F_{n32} & (\sin\beta)^{nt} & 0 & 0 & 0 \\
F_{n41} & F_{n42} & 0 & (\sin\beta)^{nt} & 0 & 0 \\
F_{n51} & F_{n52} & 0 & 0 & (\sin B)^{nt} & 0 \\
F_{n61} & F_{n62} & 0 & 0 & 0 & (\sin\beta)^{nt}
\end{bmatrix}$$

2.3 Thus we have expressed the ξ_ν as

$$\xi_1 = \sum_{n=0}^{\infty} R^{nt}(\alpha_{n1} m_1^2 \sin nt\beta + \alpha_{n2} m_1^2 \cos nt\beta + F_{n1})$$

$$\xi_2 = \sum_{n=0}^{\infty} R^{nt}(-\alpha_{n1}(1 - m_1^2)^{1/2} \cos nt\beta + \alpha_{n2}(1 - m_1^2)^{1/2} \sin nt\beta + F_{n2})$$

$$\xi_\nu = \sum_{n=0}^{\infty} R^{nt}(\alpha_{n1} F_{n\nu 1} + \alpha_{n2} F_{n\nu 2} + \alpha_{n\nu}(\sin \beta)^{nt} + F_{n\nu}), \quad \nu = 3, 4, 5, 6.$$

In this and subsequent sections we will discuss the boundary conditions for a shock attached to a wedge. This means that if the flow is assumed to follow the wedge wall, we can determine α_{n1} from the above if the wall is $\beta = 0$. The next step will be to apply the Rankine-Hugoniot equations, which can be written parametrically as $\xi_\nu = \xi_\nu(\xi_1)$ on the shock $\beta = \beta_0$.

This gives a further set of five relations to determine the $\alpha_{n\nu}$ and t. The discussion will then revolve around the conditions that must be satisfied for flows of this class to exist.

If the boundary condition at the "wall" of the wedge was instead that $\beta = 0$ was a free stream line, we would have $\xi_1(R,0) =$ constant so that α_{n2} would be determined instead of α_{n1}. The remainder of the boundary conditions would be unchanged. We will only consider the problem where the flow remains "attached" to the wall, but it should be noted that analogous results hold in this other problem.

We will discuss the problem of a wedge in a known oncoming flow in the R-β plane. The shock will be defined as $T = 0$ and the variable s will be defined along it such that $\lambda = 1$ on the shock and $s = 0$ is the stream line passing through the apex of the wedge A. In this section we will assume that the flow ahead is uniform and that the sides of the wedge are straight for a finite length. A shock is assumed to exist that will turn the flow

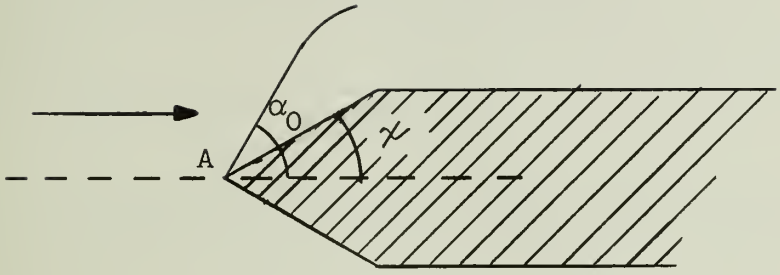

Figure 2.3

through an angle χ and be such that the flow behind the shock is subsonic at A. This can be either the strong shock or, when the angle χ is large enough, the weak shock. The boundary conditions that must be enforced at the wedge which is $\beta = 0$ are that ξ_2 is known, or in this case that

$$\xi_2 = \sum_{n=0}^{\infty} R^{nt} f_{n2}(0) = \frac{\pi}{2} + \chi$$

or

$$f_{n2}(0) = 0 \qquad n \geq 1 \ . \tag{2.3.1}$$

If we chose a particular solution $F_{n\nu}(\beta)$ such that $F_{n\nu}(0) = 0$ this implies that $\alpha_{n1} = 0 \quad n \geq 1$. This is the only condition there since the wall can support an arbitrary pressure and ξ_1 and ξ_2 were the only flow variables that were necessarily continuous at $\beta = 0$. At $\beta = \beta_0$, the shock, the R-H equations must be satisfied. These express the ξ_ν as functions of $\alpha = \xi_2 - \xi_4$. Using a Taylor series expansion we can write

$$\xi_\nu = f_{0\nu} + a_\nu(\alpha(s) - \alpha_0) + 0(\alpha(s) - \alpha_0)^2 \tag{2.3.2}$$

where the $a_\nu = \left(\frac{\partial \xi_\nu}{\partial \alpha}\right)_{\alpha_0}$ are functions of the flow ahead of the shock and α_0 only given by the R-H equations and the geometry.

Now

$$\alpha(s) - \alpha_0 = \xi_4 - \xi_2 + f_{04} - f_{02}$$

$$= R^t(f_{12}(\beta_0) - f_{14}(\beta_0)) + R^{2t}(f_{22}(\beta_0) - f_{24}(\beta_0) + \ldots \qquad (2.3.3)$$

Thus since

$$\xi_\nu(\beta_0) = \Sigma R^{nt} f_{n\nu}(\beta_0)$$

we have by substituting (2.3.3) into (2.3.2) and equating terms in R^{nt}

$$f_{n\nu}(\beta_0) = a_\nu(f_{n2}(\beta_0) - f_{n4}(\beta_0)) + Q_{n\nu} \qquad (2.3.4)$$

where $Q_{n\nu}$ is a polynomial in the $f_{m2}(\beta_0) - f_{m4}(\beta_0)$ for m = 0, 1, 2,..., n-1. Since we know $f_{0\nu}(\beta_0)$ let us assume for the inductive hypothesis that we know the $f_{m\nu}(\beta_0)$, m = 0, 1, ..., n-1, so that $Q_{n\nu}$ is known. From (2.2.5)

$$f_{n\nu}(\beta_0) = F_{n\nu}(\beta_0) + \alpha_{n\epsilon} F_{n\nu\epsilon}(\beta_0)$$

where only the $\alpha_{n\epsilon}$ are unknown, so we have on substituting into (2.3.4)

$$\alpha_{n\epsilon} F_{n\nu\epsilon}(\beta_0) + F_{n\nu}(\beta_0) = a_\nu(f_{n2}(\beta_0) - f_{n4}(\beta_0)) + Q_{n\nu}$$

or

$$\frac{\alpha_{n\epsilon} F_{n\nu\epsilon}(\beta_0) + F_{n\nu}(\beta_0) - Q_{n\nu}}{\alpha_{n\epsilon} F_{n1\epsilon}(\beta_0) + F_{n1}(\beta_0) - Q_{n1}} = \frac{a_\nu}{a_1} \qquad (2.3.5)$$

if

$$a_1 = \frac{2m_1^2 \sin 2\mu_1}{(\gamma + 1)} \neq 0 .$$

These are a set of five conditions on the flow for each power of R^t, and with the boundary conditions (2.3.1) at the wall, are the only conditions that apply in the neighborhood of the apex A of the wedge.

We will also need the following:

Lemma If the wall of the wedge is straight and the oncoming flow is uniform and if $f_{ik}(\beta_x) = 0$ for $i = 1, 2, \ldots, j$, and for $k = 1, 2$, for some β_x, then $f_{i\nu}(\beta) \equiv 0$ for $i = 1, 2, \ldots, j$, and $\nu = 1, 2, \ldots, 6$.

Proof Assume that it is true for $j = n-1$. Then from the definition of $\bar{C}_{n\nu}$ we have that $\bar{C}_{n\nu} \equiv 0$ for $\nu = 1, 2, \ldots, 6$ since the $f_{1\epsilon}, \ldots, f_{n-1\epsilon}$ are identically zero and hence a particular solution $F_{n\nu}$ is identically zero. Thus from (2.3.1) and (2.2.5) we have $\alpha_{n1} = 0$. Then since $f_{n1}(\beta_x) = f_{n2}(\beta_x) = 0$ we have from (2.2.5) that $\alpha_{n2} = 0$. The $Q_{n\nu}$ of (2.3.5) are zero when the lower order coefficients are zero as it is a polynomial in them, so we now have the denominator on the left hand side of (2.3.5) zero. Since the right hand side is finite we must have the numerator of the left hand side also zero. Thus we have

$$\alpha_{n\nu}(\sin \beta_0)^{nt} = 0 \qquad \text{for } \nu = 3, 4, 5 \text{ or } 6,$$

or

$$\alpha_{n\nu} = 0 \qquad \text{since } \sin \beta_0 \neq 0 .$$

Since the lemma is trivially true for $j = 0$ (there is nothing to prove!) we can complete the proof by induction.

Now let j be the first n such that either $f_{n1}(\beta)$ or $f_{n2}(\beta)$ is non-zero. From the lemma we know that the $f_{p\nu}(\beta)$ are identically zero for $p < j$ and so we can write from (2.2.5)

$$\begin{aligned} f_{j1} &= (\alpha_{j1} \sin t_j\beta + \alpha_{j2} \cos t_j\beta)m_1^2 \\ f_{j2} &= (-\alpha_{j1} \cos t_j\beta + \alpha_{j2} \sin tj\beta)(1 - m_1^2)^{1/2} \quad . \end{aligned} \tag{2.3.6}$$

From (2.3.1) we will have $\alpha_{j1} = 0$ and then we can substitute these into (2.3.5) to get either

$$\frac{a_2}{a_1} = \frac{(1 - m_1^2)^{1/2}}{m_1^2} \tan t_j \beta_0 \tag{2.3.7}$$

or $$\alpha_{j2} = 0 \quad .$$

Now the latter is not true for otherwise the $f_{j1}(\beta_0)$ and $f_{j2}(\beta_0)$ would both be zero contrary to the definition of j. Thus (2.3.7) defines t to be one of the countable set

$$t = \frac{1}{j\beta_0} \cot^{-1} \left\{ \frac{a_1}{a_2} \frac{(1 - m_1^2)^{1/2}}{m_1^2} \right\} + \frac{r\pi}{j\beta_0} \tag{2.3.8}$$

where r is an integer such that $t > 0$. We can choose any such r and any α_{j2} and in general get a solution.

To prove this we must show that the remainder of the boundary conditions can be satisfied. From the nature of the solutions (2.2.5) the conditions (2.3.1) can always be satisfied by choice of α_{n1}. For $\nu = 2$ (2.3.5) can be written as

$$\frac{a_2}{a_1} = \frac{(1 - m_1^2)^{1/2} \alpha_{n2} \sin t\beta_0 n + R_{n2}}{m_1^2 \alpha_{n2} \cos t \beta_0 n + R_{n1}}$$

where R_{n1} and R_{n2} are known functions of the known lower order terms and α_{n1}. This has a unique solution for α_{n2} if

$$\cot t\beta_0 n \neq \frac{a_1}{a_2} \frac{(1 - m_1^2)^{1/2}}{m_1^2} = \cot jt\beta_0 \quad .$$

Thus it has a unique solution unless

$$B_0 t(n - j) = p\pi$$

for some integer p.

This will be satisfied almost nowhere in terms of the oncoming flow, so almost everywhere there will be a unique solution for α_{n1} and α_{n2}. For $\nu = 3, 4, 5$ or 6 we have from (2.3.5)

$$\alpha_{n\nu}(\sin \beta_0)^{nt} = P_{n\nu}$$

where $P_{n\nu}$ is known in terms of α_{n1}, α_{n2} and the lower order terms, so that since $\sin \beta_0 \neq 0$, $\alpha_{n\nu}$ is determined uniquely from these lower order terms.

The flow can be analytic if jt is some integer p. Taub's[11] necessary condition that the flow is analytic is

$$A_{2j}^{p} a_j = 0 \qquad \text{for some p}$$

or that

$$\frac{a_1}{a_2} \tan p\theta \; \frac{(1 - m_1^2)^{1/2}}{m_1^2} = 1 \quad \text{at } s = T = 0 \qquad (2.3.9)$$

where

$$\tan \theta = (1 - m_1^2)^{1/2} \cot \mu_0 \quad .$$

But

$$\tan \beta_0 = (1 - m_1^2)^{1/2} \cot \mu_0 = \tan \theta \, ,$$

therefore $\beta_0 = \theta$ so that (2.3.9) is equivalent to (2.3.7) with $tj = p$. However, this condition does not imply that the flow has to be analytic for there are many choices of r and j in (2.3.8) that will make jt other than an integer.

The curvature of the shock is $\frac{\partial \alpha}{\partial s}$ which behaves asymptotically like s^{tj-1} as $s \to 0$ along the shock. Therefore the curvature will be finite but non-zero at A only if $tj = 1$. If $tj < 1$ then the curvature will be infinite at A, if $tj > 1$ it will be zero.

When (2.3.9) is satisfied with $p = 1$ the shock angle is known as the Crocco[4] angle. There we have from (2.3.8)

$$tj = 1 + \frac{r\pi}{\beta_0} \quad .$$

Since $\beta_0 < \pi/2$ and $tj > 0$, we must have $r \geq 0$, so that the curvature at A will be finite if $r = 0$ and zero if $r > 0$.

For shock angles between the Crocco angle and the extreme angle we have $tj = a + r\pi/\beta_0$ where $0 < a < 1$. Thus if $r = 0$ the shock has infinite curvature at A but if $r > 0$ it will have zero curvature there.

For shock angles between the Crocco angle and the sonic angle we will have $tj = a + r\pi/\beta_0$ where $1 < a < \pi/2$. Since $\beta_0 < \pi/2$ and r is an integer we must have $r \geq 0$. Therefore $tj > 1$ and the shock has zero curvature at A.

We have shown in this section that in general a countable choice of r and j in (2.3.8) and an arbitrary choice of α_{j2} will lead to a set of the ξ_ν which satisfy the boundary conditions at the shock and locally at the apex of a straight wedge. These parameters must in some way be determined by the boundary conditions along the sonic line that starts from the shoulder of the wedge.

We also rederived the necessary condition of Taub[11] for an analytic flow to exist and saw that this was not a sufficient condition.

2.4 Now let us consider the problem of a wedge in a known non-uniform oncoming flow.

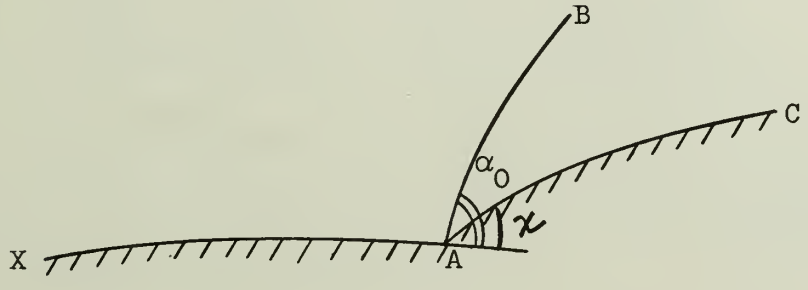

Figure 2.4

Let AC be the wedge side and XA be the known stream line passing through A. The Rankine-Hugoniot equations will give α_0 in terms of χ and the oncoming flow. This is assumed to exist and be such that the flow behind the shock at A is subsonic. We assume that the flow ahead of the shock is analytic everywhere considered, that the shock is analytic except at A, and that the flow behind the shock is analytic except at A. The suffix 0 refers to the flow ahead of the shock. Now

$$\xi_1 = (\xi_1)_0 + \frac{1}{\gamma} \log[1 + \frac{2\gamma}{\gamma + 1} (\sigma^2 - 1)]$$

so

$$\frac{\partial \xi_1}{\partial s} = \left(\frac{\partial \xi_1}{\partial s}\right)_0 + \frac{4\sigma}{\gamma + 1 + 2\gamma(\sigma^2 - 1)} \frac{\partial \sigma}{\partial s} \quad .$$

Now

$$\sigma = \frac{U_{n0}}{c_0} = \frac{v_0}{c_0} \cos(\xi_{20} - \alpha_0) \quad \text{and} \quad U_{t0} = v_0 \sin(\xi_{20} - \alpha_0)$$

therefore

$$\frac{\partial \sigma}{\partial s} = k \frac{U_{t0}}{c_0} - \frac{U_{t0}}{c_0} \left(\frac{\partial \xi_2}{\partial s}\right)_0 + \frac{U_{n0}}{v_0 c_0} \frac{\partial v_0}{\partial s} - \frac{U_{n0}}{c_0^2} \frac{\partial c_0}{\partial s}$$

hence

$$\frac{\partial \xi_1}{\partial s} = k a_1 + B_1 \qquad (2.4.1)$$

where B_1 is a function of the flow variables ahead of the shock,

$$\left(\frac{\partial \xi_2}{\partial s}\right)_0, \left(\frac{\partial c_1}{\partial s}\right)_0, \frac{\partial v_0}{\partial s}, \text{ and } \left(\frac{\partial \xi_1}{\partial s}\right)_0$$

only, and so is dependent on the shock direction but not its derivatives. If Φ is the rotation ahead of the shock, and L the specific enthalpy we have (see Taub[11] equations (4.17b) and (4.19a))

$$U_{n0}\Phi = v_0 \frac{\partial v_0}{\partial s} + c_0^2 \left(\frac{\partial \xi_1}{\partial s}\right)_0; \quad h_{0,s} = U_{n0}\Phi - c_0^2\left(\frac{\partial \xi_1}{\partial s}\right)_0 + \frac{2c_0}{\gamma-1}\frac{\partial c_0}{\partial s}$$

so that

$$B_1 = \left(\frac{\partial \xi_1}{\partial s}\right)_0 + \frac{4\sigma}{\gamma+1+2\gamma(\sigma^2-1)}\left[-\frac{U_{t0}}{c_0}\left(\frac{\partial \xi_2}{\partial s}\right)_0 + \frac{U_{n0}}{v_0^2 c_0}\left\{U_{n0}\Phi - c_0^2\left(\frac{\partial \xi_1}{\partial s}\right)_0 - \frac{U_{n0}}{2c_0^3}(\gamma-1)\left\{h_{0,s} - U_{n0}\Phi + c_0^2\left(\frac{\partial \xi_1}{\partial s}\right)_0\right\}\right]\right] \tag{2.4.2}$$

Now

$$\frac{\partial \xi_2}{\partial s} = k + \frac{\partial \mu}{\partial s} = k + \frac{1}{v^2}\left[U_n \frac{\partial U_t}{\partial s} - U_t \frac{\partial U_n}{\partial s}\right]$$

$$= ka_2 + B_2 \tag{2.4.3}$$

where B_2 is also a function of the flow ahead and the shock direction only.

We look for a solution of the form (2.2.5) behind the shock. At $\beta = 0$, the wedge, we have

$$\xi_2(T,0) = 0 = \sum_{n=0}^{\infty} T^{nt} f_{n2}(0) .$$

Let us assume that the shape of the wedge is a known analytic function of T. Then we must have $t = 1/p$ for some integer $p > 0$ and $f_{n2}(0) = 0$, unless $p \mid n$. Since $1 \le \sigma^2 \le m_0^2$, $a_2 \ge \frac{\gamma-1}{\gamma+1} > 0$. Thus we get from (2.1.4), (2.4.1) and (2.4.3)

$$\frac{\frac{\partial \xi_1}{\partial s} - B_1}{\frac{\partial \xi_2}{\partial s} - B_2} = \frac{ka_1}{ka_2} = \frac{a_1}{a_2} \qquad \text{at} \qquad T = 0 \quad \text{if } k \neq 0$$

or that

$$\frac{\sum R^{n/p-1}\ \frac{n}{p}\ f_{n1}(\beta_0)(1 - m_1^2\sin^2\alpha_0)^{1/2} - B_1}{\sum R^{n/p-1}\ \frac{n}{p}\ f_{n2}(\beta_0)(1 - m_1^2\sin^2\alpha_0)^{1/2} - B_2} = \frac{a_1}{a_2} \quad .$$

Taking the limit as s or $R \longrightarrow 0$ and assuming $B_1,\ B_2 \neq 0$

$$\frac{a_1}{a_2} = \frac{B_1}{B_2} \quad \text{if} \quad \frac{j}{p} > 1 \ , \qquad \text{(a)}$$

$$\frac{a_1}{a_2} = \frac{B_1 - f_{j1}(\beta_0)(1 - m_1^2\sin^2\alpha_0)^{1/2}}{B_2 - f_{j2}(\beta_0)(1 - m_1^2\sin^2\alpha_0)^{1/2}} \quad \text{if} \quad \frac{j}{p} = 1 \qquad \text{(b)} \qquad (2.4.4)$$

or

$$\frac{a_1}{a_2} = \frac{f_{j1}(\beta_0)}{f_{j2}(\beta_0)} \quad \text{if} \quad \frac{j}{p} < 1 \qquad \text{(c)}$$

where j is the first n such that either $f_{n1}(\beta_0)$ or $f_{n2}(\beta_0)$ is non-zero. Now if $j < p$ then we have from the lemma of section 2.3, $f_{i\nu}(\beta) = 0$ for $i < j$, so that

$$f_{j1} = m_1^2(\alpha_{j1} \sin \frac{j}{p}\beta + \alpha_{j2} \cos \frac{j\beta}{p})$$

$$f_{j2} = (1 - m_1^2)^{1/2}(-\alpha_{j1} \cos \frac{j}{p}\beta + \alpha_{j2} \sin \frac{j\beta}{p}) \quad .$$

But

$$f_{j2}(0) = 0, \qquad \text{so} \quad \alpha_{j1} = 0,$$

so from (2.4.4)

$$\frac{a_1}{a_2} = \frac{m_1^2}{(1 - m_1^2)^{1/2}} \cot (\frac{j}{p}\beta_0) \qquad (2.4.5)$$

Thus if there exists a positive rational number $r < 1$ such that

$$\cot(r\beta_0) = \frac{a_1}{a_2} \frac{(1 - m_1^2)^{1/2}}{m_1^2}$$

there can exist a solution of this type. However this is not in general true so we must have $j \geq p$. Then we can group cases (a) and (b) of (2.4.4) together to get

$$\frac{a_1}{a_2} = \frac{B_1(1 - m_1^2\sin^2\alpha_0)^{-1/2} - f_{p1}(\beta_0)}{B_2(1 - m_1^2\sin^2\alpha_0)^{-1/2} - f_{p2}(\beta_0)} \quad .$$

We will have

$$f_{p1} = m_1^2(\alpha_{p1}\sin\beta + \alpha_{p2}\cos\beta)$$

$$f_{p2} = (1 - m_1^2)^{1/2}(-\alpha_{p1}\cos\beta + \alpha_{p2}\sin\beta)$$

and

$$\alpha_{p1} = -K_2(1 - m_1^2)^{-1/2}$$

where

$$K_2 = \left(\frac{\partial \mathfrak{F}_2}{\partial T}\right)_{\beta=0,R=0}$$

so α_{p2} is the solution of

$$\frac{a_1}{a_2} = \frac{B_1(1 - m_1^2\sin^2\alpha_0)^{-1/2} - \alpha_{p2}m_1^2\cos\beta_0 + K_2m_1^2\sin\beta_0(1 - m_1^2)^{-1/2}}{B_2(1 - m_1^2\sin^2\alpha_0)^{-1/2} - \alpha_{p2}(1 - m_1^2)^{1/2}\sin\beta_0 - K_2\cos\beta_0} \quad . \tag{2.4.6}$$

This exists and is unique if

$$\frac{a_1}{a_2} \neq \frac{m_1^2}{(1 - m_1^2)^{1/2}} \cot\beta_0 \quad . \tag{2.4.7}$$

This is violated at the Crocco[4] angle, so that at the wedge angle which, in the case of a straight wedge, allows an analytic solution we have no solution

to this more general problem unless (2.4.6) is satisfied when $\alpha_{p2} = 0$. (2.4.7) is violated again when $\gamma = 0$ which implies that $\alpha_0 = \pi/2 = \beta_0$ and that $a_1 = 0$. Then we must have

$$B_1 = -K_2 m_1^2 \quad . \tag{2.4.8}$$

If, in addition, flow ahead is irrotational and is isenthalpic, then we have from (2.4.2)

$$B_1 = \left(\frac{\partial \xi_1}{\partial s}\right)_0 - \frac{4m_0}{\gamma + 1 + 2\gamma(m_0^2 - 1)} \left[U_{n0} \frac{\gamma - 1}{2c_0} \left(\frac{\partial \xi_1}{\partial s}\right)_0 + \frac{U_{n0} c_0}{v_0^2} \left(\frac{\partial \xi_1}{\partial s}\right)_0 \right]$$

or

$$B_1 = -m_0^2 \left(\frac{\partial \xi_2}{\partial T}\right)_0 \frac{2m_0^2 - (\gamma + 3)}{2\gamma m_0^2 - (\gamma - 1)} \quad .$$

Then (2.4.8) is precisely the condition of Tsien[13]. If (2.4.8) is not satisfied then a solution of this class does not exist.

2.5 Lin and Rubinov[8] discuss a solution which relaxes Rsien's condition to an inequality when a shock involving a singularity of the type $S^{1/2}$ is used. We have seen that it is not one of the family of solutions discussed and I believe that such a solution does not exist, so let us discuss their technique for finding a solution. Similar remarks apply to the work of Chow[3] in his extension of the results to the psuedo-stationary problem. For the normal shock problem let us suppose we know

$$\xi_2 - \mu = \alpha = \frac{\pi}{2} + bs^{\delta} + o(s^{\delta}) \quad \text{for } \delta > 0$$

where b and δ are parameters to be determined. We can then calculate

$$\frac{\partial \xi_1}{\partial s}(s,0) = B_1 - \frac{4\delta m_0^2 b^2 s^{2\delta - 1}}{\gamma + 1 + 2\gamma(m_0^2 - 1)} + o(s^{2\delta - 1})$$

$$\frac{\partial \xi_2}{\partial s}(s,0) = B_2 + \frac{\delta 2bs^{\delta-1}(1 - m_0^2)}{2 + (\gamma - 1)m_0^2} + o(s^{\delta-1}) .$$

Now we can write

$$\xi_i(s,T) = \xi_i(s,0) + T \frac{\partial \xi_i}{\partial T}(s,0) + O(T^2)$$

$$= \xi_i(s,0) + TA_{ij} \frac{\partial \xi_j}{\partial s}(s,0) + O(T^2)$$

where

$$(A_{ij})_{s=0} = (1 - m_1^2)^{-1} \begin{bmatrix} 0 & m_1^2 \\ 1 - \frac{1}{m_1^2} & 0 \end{bmatrix}$$

in this case. Therefore

$$\xi_2(0,T) = \frac{\pi}{2} + \frac{T}{m_1^2} \left\{ \frac{4\delta m_0^2 b^2}{\gamma + 1 + 2\gamma(m_0^2 - 1)} \lim_{s \to 0} s^{2\delta-1} - B_1 \right\} + O(T^2) . \tag{2.5.1}$$

Since

$$\xi_2(0,T) = \frac{\pi}{2} + K_2 T + O(T^2)$$

we must have either $\delta = 1/2$ or $B_1 = -K_2 m_1^2$ so that $\lim_{s \to 0} s^{2\delta-1}$ is finite and non-zero. Thus

$$K_2 + \frac{B_1}{m_1^2} = \frac{1}{m_1^2} \frac{2m_0^2 b^2}{\gamma + 1 + 2\gamma(m_0^2 - 1)} \quad \text{or } 0$$

which gives the Lin and Rubinov condition that

$$K_2 m_1^2 + B_1 \geq 0 . \tag{2.5.2}$$

Since a shock that behaves like $s^{1/2}$ around $s = 0$ requires that $j/p = 1/2$ in the treatment of section 2.4 we must satisfy equation (2.4.5) if the Lin and Rubinov solution is to be a member of the family of solutions discussed. However

since $a_1 = 0$ and $\beta_0 = \pi/2$, (2.4.5) is not satisfied. Therefore I do not believe that a solution of the type they suggest exists. The false reasoning in the above outline of the argument must be that the error term $O(T^2)$ of equation (2.5.1) is not bounded. This can happen since it was the limit as s tended to zero of an earlier term which may behave like $s^{-1/2}$, and so become infinite.

Now the error term $O(T^2)$ can be expressed as

$$\frac{1}{2} T^2 \frac{\partial^2 \zeta_2}{\partial T^2} (s,\theta) \qquad \text{for some } \theta \in (0,T) .$$

We cannot evaluate this explicitly, nor get a lower bound for it, but we can show it tends to ∞ at $\theta = 0$ as s tends to 0.

$$\frac{\partial^2 \zeta_2}{\partial T^2} = - \frac{1}{1 - m_1^2} \frac{\partial^2 \zeta_2}{\partial s^2} + O(s^{2\delta-1}) \quad \text{at} \quad T = 0$$

$$= - \frac{1}{1 - m_1^2} \frac{s^{\delta-2} \delta(\delta - 1)\, 2b(1 - m_0^2)}{(\gamma - 1) m_0^2 + 2} + O(s^{2\delta-1})$$

which $\to \infty$ as $s \longrightarrow 0$ for $\delta = 1/2$. Consequently we have no guarantee that (2.5.1) is a representation of ζ_2 at s = 0.

If we estimate the radius of convergence of ζ_2 expressed as a power series in T about the point (s,0) we will have

$$\zeta_2(s,T) = \sum_{n=0}^{\infty} \frac{T^n}{n!} \frac{\partial^n \zeta_2}{\partial T^n} (x,0) .$$

The radius of convergence is

$$\left[\limsup_{n \to \infty} \left| \frac{1}{n!} \frac{\partial^n \zeta_2}{\partial T^n} \right|^{1/n} \right]^{-1} .$$

For n even we have

$$\frac{\partial^n \zeta_2}{\partial T^n} = A^n_{2j} \frac{\partial^n \zeta_j}{\partial s^n} + Q_{n2}$$

where $Q_{n2} = o(s^{\delta-n})$

$$\frac{\partial^n \zeta_2}{\partial s^n} = \delta^{(n)} \frac{s^{\delta-n} 2b(1 - m_0^2)}{(\gamma - 1) m_0^2 + 2} + o(s^{\delta-n})$$

where $\delta^{(n)} = \delta(\delta - 1) \cdots (\delta - n + 1)$. Therefore

$$\frac{\partial^n \zeta_2}{\partial T^n} = \frac{\delta^{(n)} 2b\, s^{\delta-n}(1 - m_0^2)}{(\gamma - 1)m_0^2 + 2} \left(\frac{1}{m_1^2 - 1}\right)^{n/2} + o(s^{\delta-n}) .$$

Therefore the radius of convergence may be

$$\leq \left[\limsup_{n\to\infty} \left| \frac{\delta^{(n)}}{n!} \right|^{1/n} |s|^{\frac{\delta-n}{n}} \left(\frac{1}{1 - m_1^2}\right)^{1/2} \right]^{-1}$$

$$= (1 - m_1^2)^{1/2} |s|$$

if δ is not an integer. Thus the region of convergence of these power series may be as given in Figure 2.5 in the neighborhood of A.

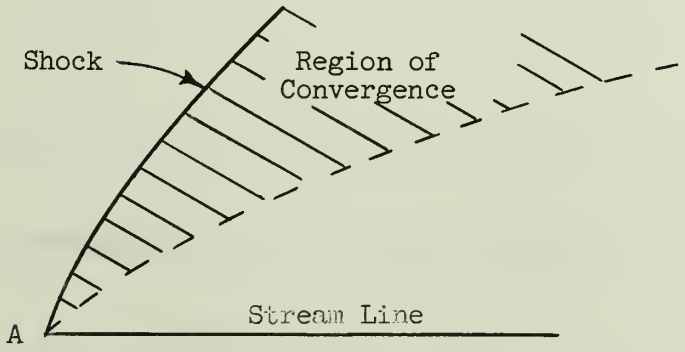

Figure 2.5

2.6 A note on pseudo-stationary flow.

The equations of psuedo-stationary flow can be written in the form (see Taub[11'])

$$\frac{\partial \xi_1}{\partial s} = \xi_5 \sin \xi_4 \frac{\partial \xi_1}{\partial T} - \xi_5 \xi_3 \cos \xi_4 \frac{\partial \xi_2}{\partial T}$$

$$\frac{\partial \xi_2}{\partial s} = \xi_5 \left(\frac{1}{\xi_3} - 1 \right) \cos \xi_4 \frac{\partial \xi_1}{\partial T} + \xi_5 \sin \xi_4 \frac{\partial \xi_2}{\partial T} - \frac{\xi_5}{\xi_6} \cos \xi_4$$

$$\frac{\partial \xi_3}{\partial T} = -[2 + (\gamma - 1)\xi_3] \frac{\partial \xi_1}{\partial T} - \frac{\xi_3}{\xi_6}$$

$$\frac{\partial \xi_4}{\partial T} = \frac{\partial \xi_2}{\partial T} - \frac{\sin \xi_4}{\xi_5} \frac{\partial \xi_2}{\partial s}$$

$$\frac{\partial \xi_5}{\partial T} = \cos \xi_4 \frac{\partial \xi_2}{\partial s}$$

$$\frac{\partial \xi_6}{\partial T} = -\frac{\xi_6}{\xi_3} \frac{\partial \xi_1}{\partial T} - 1 \qquad (2.6.1)$$

or if $||a_{ij}||$ can be inverted as

$$\frac{\partial \xi_\nu}{\partial T} = A_{\nu j} \frac{\partial \xi_j}{\partial s} + H_\nu \qquad (2.6.2)$$

where

$$H_\nu = \left[-A_{12} \frac{\xi_5}{\xi_6} \cos \xi_4, \; -A_{22} \frac{\xi_5}{\xi_6} \cos \xi_4, \; -\frac{\xi_3}{\xi_6} - H_1(2 + (\gamma - 1)\xi_3), \; H_2, \; 0, \; -1 - H_1 \frac{\xi_6}{\xi_3} \right].$$

We will make the transformation to the R-β plane and substitute

$$\xi_\nu = \sum_{n=0}^{\infty} R^{nt} f_n(\beta)$$

as before. H_ν which is a function of the ξ_ϵ can be written as

$$H_\nu = \sum_{q=0}^{\infty} H_{q\nu}(\beta)\, R^{nt}$$

where the $H_{q\nu}$ are functions of the $f_{m\epsilon}$ for m = 0, 1, 2, ..., q. The resulting differential equations will be

$$\sum_n R^{tn-1}(\dot{f}_{n\nu} - D_{\nu\epsilon}\, f_{n\epsilon} nt - \bar{C}_{n\nu}) = \sum_q E^{-1}_{\nu\epsilon} H_{q\epsilon}(\beta)\, R^{qt} \quad . \tag{2.6.3}$$

In order to match terms we must require that for each $q \geq 0$ there exists an n such that $nt-1 = qt$. Thus t is the reciprocal of an integer p. If we substitute for this t into (2.6.3) and equate terms in R^{nt-1} we will get

$$\dot{f}_{n\nu} = \frac{n}{p} D_{\nu\epsilon} f_{n\epsilon} + \bar{J}_{n\nu} \tag{2.6.4}$$

where

$$\bar{J}_{n\nu} = \bar{C}_{n\nu} \qquad 0 \leq n < p$$

$$= \bar{C}_{n\nu} + E^{-1}_{\nu\epsilon} H_{n-p\epsilon} \qquad p \leq n \quad .$$

In particular

$$E^{-1}_{\nu\epsilon} H_{0\epsilon} = \left[- \frac{m_1 \sin\beta}{c_1(1 - m_1^2)} \;,\; \frac{-\cos\beta}{m_1 c_1 (1 - m_1^2)^{1/2}} \;,\; x,\; x,\; x,\; x \right] \tag{2.6.5}$$

so that if $\bar{C}_{p\nu}$ is zero, a particular solution of (2.6.4) for $n = p$ is

$$F_{p\nu}(\beta) = \left[0,\; \frac{-\sin\beta}{m_1 c_1 (1 - m_1^2)^{1/2}} \;,\; x,\; x,\; x,\; x \right] \quad . \tag{2.6.6}$$

We will prove that if a solution for r of

$$\cot r\beta_0 = \frac{a_1}{a_2} \frac{(1 - m_1^2)^{1/2}}{m_1^2} \tag{2.6.7}$$

is a positive rational number, then a solution to the local boundary conditions in regular reflection can be written in the form

$$\xi_\nu = \sum_{n=0}^{\infty} R^{n/p} f_{n\nu}(\beta) \qquad \text{where} \quad r = \frac{j}{p} \quad .$$

In particular the first n such that all of the $f_{n\nu}(\beta)$ are not identically zero is $n = \min(j,p)$.

First note that Taub[11] has shown that an analytic solution exists if (2.6.7) admits an integer solution. Since the class of functions we are considering includes the analytic functions, it is already proved for integral r, so we only have to consider those r which can be written as j/p where $p \nmid j$ (p does not divide j). The problem of regular reflection is equivalent to the

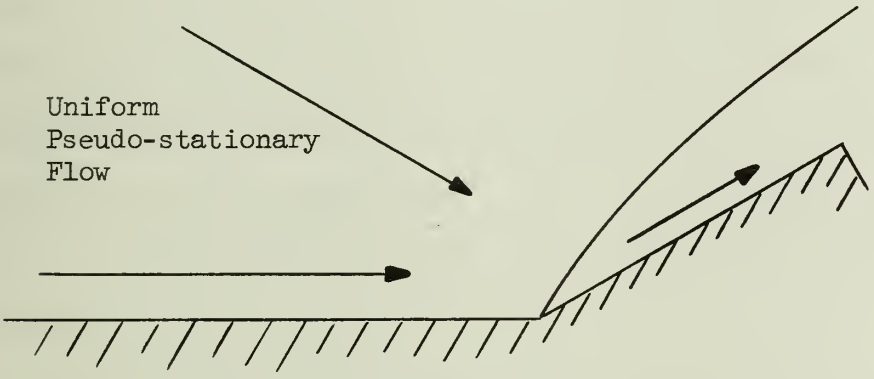

Figure 2.6

problem of a wedge in a uniform pseudo-stationary oncoming flow, so we will consider this latter problem only. The flow behind the shock at the apex of the wedge is assumed subsonic.

The boundary conditions at the wall behind the shock are that $f_{n2}(0) = 0$ for $n \geq 1$. At the shock we have (see Taub[11] equations 8.2 and 8.8)

$$\frac{\partial \xi_1}{\partial s} = ka_1, \qquad \frac{\partial \xi_2}{\partial s} - \frac{\cos \xi_4}{\xi_6}$$

and we can express the other conditions similarly as

$$\frac{\partial \xi_\nu}{\partial s} = k(s)\, a_\nu(s) + b_\nu(s) \tag{2.6.8}$$

where $a_\nu(s)$ and $b_\nu(s)$ are functions of the $\xi_\epsilon(s)$ but not their derivatives. Now $k(s)R$ can be expressed as

$$C_1 R^t + C_2 R^{2t} + \cdots$$

and both $a_\nu(s)$ and $b_\nu(s)$ can be expressed in the form

$$a_\nu + a_{\nu 1} R^t + a_{\nu 2} R^{2t} + \cdots$$

where the C_m, $a_{\nu m}$, and $b_{\nu m}$ are functions of the $f_{i\epsilon}$ $i = 0, 1, \ldots, m$ only. Further we note that the $a_{\nu m}$, C_m and $b_{\nu m}$ are zero if $p \nmid m$ and if all the $f_{i\epsilon}(\beta)$ are identically zero for all $1 \le i \le m$ such that $p \nmid i$. This happens because the $a_{\nu m} R^{mt}$, $C_m R^{mt}$ and the $b_{\nu m} R^{mt}$ are sums of products of the $f_{n\epsilon} R^{nt}$ with coefficients which depend on the $f_{0\epsilon}$ only, so that they can be written as sums of terms like $\prod_i f_{m_i \epsilon_i}(\beta)$ where $\sum_i m_i = m$. Each term in the sum is thus zero unless p divides each m_i. However this implies that p divides m contrary to hypothesis.

Thus if we take the terms in R^{nt-1} from (2.6.8) we will get equations of the form

$$\frac{f_{n\nu}}{f_{n1}} = \frac{a_{\nu 0} C_n + R_{n\nu}}{a_{10} C_n + R_{n1}} \tag{2.6.9}$$

where

$$R_{n\nu} = \sum_{q=1}^{n-1} C_q a_{\nu\, n-q} + b_{\nu\, n-p}$$

and thus are functions of the lower order $f_{m\epsilon}$. Therefore the $R_{n\nu}$ are also zero if $p \nmid n$ and if all of the $f_{m\epsilon}$ are identically zero for $1 \le m \le n-1$ such that $p \nmid m$.

Returning to the differential equations (2.7.4) we note that by the same reasoning the $\bar{J}_{n\nu}$ are identically zero under the same conditions that the $R_{n\nu}$ are zero.

Now we are in a position to set up the inductive hypothesis. Suppose all of the $f_{m\epsilon}$ are identically zero for $m \leq n - 1$ such that $p \nmid m$ and $p \nmid n$. Then if $n < j$, the $f_{n\nu}$ are identically zero.

Note that the $R_{n\nu}$ and $\bar{J}_{n\nu}$ are both zero under the conditions of the hypothesis so that a particular solution of the differential equations (2.6.4) is zero. In the general solution (2.2.5) we have that $\alpha_{n1} = 0$ from the condition at the wall of the wedge. Then from (2.6.9) we have

$$\frac{a_2}{a_1} = \frac{(1 - m_1^2)^{1/2}}{m_1^2} \quad \frac{\alpha_{n2} \sin n \beta_0/p}{\alpha_{n2} \cos n \beta_0/p} \tag{2.6.10}$$

so that if $n \neq j$, $\alpha_{n2} = 0$. Then the denominator of the left hand side of (2.6.9) is zero, so that since the right hand side is well-behaved, the numerator of the left hand side will also be zero. On substituting the general solution from (2.2.5) we will get $\alpha_{n\nu} = 0$ for $\nu = 3, 4, 5$ and 6. Thus we have shown that the $\alpha_{n\nu}$ are identically zero.

Thus by induction we can show that the $f_{n\nu}(\beta)$ are identically zero for all $n < j$ such that $p \nmid n$. When $n = j$ (2.6.10) admits an arbitrary solution for α_{j2} since $r = j/p$ satisfies (2.6.7). For $n > j$ there will be extra terms in the equations (2.6.10) so these will then have a non-zero solution for the α_{n2} in general.

The cases that are still unsolved are thus those where there does not exist a positive rational solution of (2.6.7). A method for finding a solution in these cases is to express ξ_ν as $\sum_n \sum_m R^{n+nt} f_{nm\nu}(\beta)$. Substituting into (2.6.2) we get

$$\dot{f}_{nm\nu} = (n + mt)f_{nm\epsilon}D_{\nu\epsilon} + \bar{C}_{nm\nu}$$

where the matrix D is as before and the $\bar{C}_{nm\nu}$ are polynomials in the $f_{rs\epsilon}$ and the $\dot{f}_{rs\epsilon}$ for r = 0, 1, ..., n and for s = 0, 1, ..., m, except that they do not include the f_{mn} or $\dot{f}_{mn}$. In particular

$$\bar{C}_{00\nu} = 0$$

$$\bar{C}_{01\nu} = 0$$

and $\bar{C}_{10}$ is the $E^{-1}_{\nu\epsilon}\ H_{0\epsilon}$ given by (2.6.5). By techniques similar to the treatment of stationary flow we can write the solutions as

$$\zeta_1 = f_{001} + m_1^2(\alpha_{101}\sin\beta + \alpha_{102}\cos\beta)R + m_1^2(\alpha_{011}\sin t\beta + \alpha_{012}\cos t\beta)R^t + \ldots$$

$$\zeta_2 = f_{002} + (1 - m_1^2)^{1/2}(-\alpha_{101}\cos\beta + \alpha_{102}\sin\beta)R$$

$$+ (1 - m_1^2)^{1/2}(-\alpha_{011}\cos t\beta + \alpha_{012}\sin t\beta)R^t - \frac{R\sin\beta}{m_1c_1(1 - m_1^2)^{1/2}} + \ldots \qquad (2.6.11)$$

Since $\frac{\partial\zeta_2}{\partial T} = 0$ at $\beta = 0$ we have $\alpha_{101} = \alpha_{011} = 0$. From the R-H equations we have

$$\zeta_1(R,\beta_0) = f_{001} + a_1(\alpha(s) - \alpha_0) + 0(\alpha(s) - \alpha_0)^2$$

and

$$\zeta_2(R,\beta_0) = f_{002} + a_2(\alpha(s) - \alpha_0) - \frac{\cos\mu_1}{m_1c_1}(1 - m_1^2\cos^2\mu_1)^{-1/2}R + \ldots$$

These can be obtained directly by integrating equations (2.6.8) and the fact that $R = s(1 - m_1^2\cos^2\mu_1)^{1/2}$ at $\beta = \beta_0$. Substituting these into equations (2.6.11) and using the fact that

$$\alpha(s) - \alpha_0 = (f_{012}(\beta_0) - f_{014}(\beta_0))R^t + (f_{102}(\beta_0) - f_{104}(\beta_0))\, R + \ldots$$

we get on equating terms in R^t and R

$$\frac{a_2}{a_1} m_1^2 \alpha_{012} \cos t\beta_0 = a_2(f_{012}(\beta_0) - f_{014}(\beta_0)) = (1 - m_1^2)^{1/2} \alpha_{012} \sin t\beta_0 \tag{2.6.12}$$

and

$$\frac{a_2}{a_1} m_1^2 \alpha_{102} \cos \beta_0 = a_2(f_{102}(\beta_0) - f_{104}(\beta_0)) = (1 - m_1^2)^{1/2} \alpha_{102} \cos \beta_0 \tag{2.6.13}$$

since

$$\frac{\cos \mu_1}{(1 - m_1^2 \cos^2 \mu_1)^{1/2}} = \frac{\cot \mu_1}{(1 + (1 - m_1^2)\cot^2 \mu_1)^{1/2}} = \frac{\tan \beta_0 (1-m_1^2)^{1/2}}{\sec \beta_0} = \frac{\sin \beta_0}{(1-m_1^2)^{1/2}}$$

Since we are assuming that $\frac{1}{\beta_0} \cot^{-1}\left\{ \frac{a_1}{a_2} \frac{(1 - m_1^2)^{1/2}}{m_1^2} \right\}$ is not an integer we have $\alpha_{102} = 0$ from (2.6.13). t is given by (2.6.12) and α_{012} is arbitrary.

Thus once again we have a coutable choice of t and an arbitrary parameter to choose, both of which must be determined in some way by additional boundary conditions. The remaining parameters will be determined uniquely in terms of these provided that

$$\cot[(mt + n)\beta_0] \neq \frac{a_1}{a_2} \frac{(1 - m_1^2)^{1/2}}{m_1^2} = \cot t\beta_0$$

for all n, $m \geq 0$ except for $n = 0$ and $m = 1$. Again we can say that this happens for almost all oncoming Mach numbers and wedge angles since it requires that

$$\beta_0(n - 1)t + m \neq p\pi \tag{2.6.14}$$

for some integer p.

2.7 We can raise the question of what happens if $\frac{1}{\beta_0}\cot^{-1}\left\{\frac{a_1}{a_2}\frac{(1-m_1^2)^{1/2}}{m_1^2}\right\}$ is a rational number and we use a double series? A more general question is what would happen if we used a multiple series of the form

$$\zeta_\nu = \sum_{n1}\sum_{n2}\cdots\sum_{nk} R^{n,t_1+\ldots+n_kt_k} f_{n,\ldots n_k}(\beta_0) \qquad (2.7.1)$$

in the stationary case and of the form

$$\zeta_\nu = \sum_{n1}\cdots\sum_{nk}\sum_{m} R^{n,t_1+\ldots+n_kt_k+m} f_{n,\ldots n_k m}(\beta_0) \qquad (2.7.2)$$

in the pseudo-stationary case? If we follow the techniques of section 2.6 we will get a number of equations of the form of (2.6.12) and (2.6.13) each giving rise to an arbitrary parameter $\alpha_{\ldots 2}$ if the t_i is a solution of

$$\cot t_i\beta_0 = \frac{a_1}{a_2}\frac{(1-m_1^2)^{1/2}}{m_1^2} .$$

Thus we can have a solution of this form with k arbitrary parameters if the higher order boundary conditions can be satisfied. They can if there do not exist positive integers n_i, at least two of them being non-zero, such that

$$n,t_1 + \ldots + n_kt_k = t_1 + \frac{\pi p}{\beta_0} \qquad (2.7.3)$$

in the stationary case. This will not be satisfied in general since we can write $t_i = t_1 + \frac{\pi}{\beta_0} q_i$ where q_i is an integer, so that (2.8.3) is the condition that

$$(n_1 + n_2 + \ldots + n_k)t_1\beta_0 - t_1\beta_0 = (p - n_2q_2\ldots n_kq_n)\pi. \qquad (2.7.4)$$

$\pi/\beta_0 t_1$ is not a rational number in general so that this does not admit a solution with $n_1 + n_2 + \ldots + n_k \geq 2$. In the pseudo-stationary case the equivalent

condition is that

$$(n_1 + n_2 \ldots + n_k)t_1\beta_0 - t_1\beta_0 + m\beta_0 = m\beta_o = \pi(p - n_2q_2 \ldots n_kq_k) \quad (2.7.5)$$

which is not satisfied in general.

Thus there are equivalent solutions to the stationary and pseudo-stationary problems of the form (2.7.1) and (2.7.2) respectively, where the t_i satisfy

$$\cot t_i\beta_o = \frac{a_1}{a_2} \frac{(1 - m_1^2)^{1/2}}{m_1^2} .$$

There are k arbitrary parameters provided that equations (2.7.4) and (2.7.5) cannot be satisfied in the respective cases.

In conclusion it is worth noting that there are numerous techniques that could be applied to this and related problems. For example in the preceeding a particular class of orthogonal functions of the independent variable R was used, namely a set of different powers of R. If $\{\phi_n(R);\ n = 0, 1, \ldots\}$ are a set of orthogonal functions of R such that

$$\frac{d\phi_n}{dR}(R) \equiv F_n(\phi_0(R), \ldots \phi_n(R))$$

and

$$\phi_n(R)\phi_m(R) \equiv G_n(\phi_0(R), \ldots \phi_{n+m}(R))$$

where the F_n and G_n are linear functions of the ϕ's, then the use of a series $\sum_{n=0}^{\infty} \phi_n(R)f_{n\nu}(\beta)$ for the ξ_ν would result in a set of linear total differential equations in the $f_{n\nu}(\beta)$ whose coefficients are functions of the $f_{0\nu}(\beta)$, ... $f_{n-1\nu}(\beta)$ only. We could ask what set of orthogonal polynomials lead to the simplest solution, or if any other set will yield a solution to some of the remaining unsolved problems. More generally we could ask are there suitable

independent variables with which to use this technique in partial differential equations of the type

$$\frac{\partial \xi_\nu}{\partial s} = A_{\nu\epsilon} \frac{\partial \xi_\epsilon}{\partial T}$$

for arbitrary matrices $A_{\nu\epsilon}$?

III VISCOUS FLOW

3.1 The algebraic R-H equations and the algebraic matching conditions across non-viscous shocks and slip streams only admit trivial solutions when applied around the triple point in Mach reflection and no solution when applied to regular reflection if the incident shock is weak and the angle of the incident shock close to the angle at which the flow behind the shock is sonic. Experimentally, however, both regular and Mach reflection can be observed in this region (see Smith[9]). Since the techniques of Chapter II required that shocks exist that turn the flow through the appropriate angles, those techniques cannot be used in discussions of Mach reflection and regular reflection for which no solution to the R-H equations or no non-trivial solutions to these equations exist.

The incident shock being weak implies that the ratio of the pressure on either side of the shock is close to 1 or that

$$\frac{p_1}{p_0} = 1 + \frac{2\gamma}{\gamma+1}(\sigma^2 - 1) \approx 1$$

so that $\sigma^2 \approx 1$. Then if the flow behind the shock is nearly sonic we have from (1.2.1) - (1.2.4)

$$m_1^2 = \frac{(m_0^2 - \sigma^2)(\gamma+1)^2\sigma^2 + ((\gamma-1)\sigma^2 + 2)^2}{(2 + (\gamma-1)\sigma^2)(2\gamma\sigma^2 + 1 - \gamma)} \approx 1 .$$

Since $\sigma^2 \approx 1$ we get

$$1 \approx m_1^2 \approx m_0^2$$

so that the cases of greatest interest are those for which the Mach number of the oncoming flow is close to 1.

We are going to try to modify the R-H equations and the matching conditions across stream lines by introducing viscosity and then considering the result as viscosity tends to zero. The equations of viscous flow are given by (1.1.1) through (1.1.3). We assume that the viscosity w is a constant. If we replace the line integrals in (1.1.1) - (1.1.3) by area integrals of the divergence we can then express the equations as second order partial differential equations in four unknowns. These are too complicated to be solved analytically and until more is known of the appropriate boundary conditions for a given problem it is difficult to see how to integrate them directly numerically.

We can transform the independent variables x_1 and x_2 of equations (1.1.1) - (1.1.3) into the s-T coordinates defined in section 1.2 to get

$$\int_{T_0}^{T_1}\left[\int_{s_0}^{s_1} \rho U_n ds\right] = 0 \qquad (3.1.1)$$

$$\int_{T_0}^{T_1}\left[\int_{s_0}^{s_1} (\rho U_n u_i + \lambda_j \epsilon_{ij} p + wH_i) ds\right] - \left[\int_{s_0}^{s_1}\int_{T_0}^{T_1} (p \frac{u_j}{v} \epsilon_{ij} + wG_i) dT\right] = 0 \qquad (3.1.2)$$

and

$$\int_{T_0}^{T_1}\left[\int_{s_0}^{s_1} (\rho U_n \frac{v^2}{2} + \frac{\gamma}{\gamma - 1} pU_n + wI) ds\right] + \left[\int_{s_0}^{s_1}\int_{T_0}^{T_1} wJ dT\right] = 0 \qquad (3.1.3)$$

where

$$G_i = -\frac{2\epsilon_{ij}u_j}{3U_n}\left(\lambda \cos\mu \frac{\partial v}{\partial T} - U_t \frac{\partial \xi_2}{\partial T} + v \frac{\partial \xi_2}{\partial s}\right)$$

$$+ \frac{1}{\lambda \cos\mu}\left(\frac{u_i}{v}\lambda \sin\mu \frac{\partial v}{\partial T} - \frac{u_i}{v}\frac{\partial v}{\partial s} - \epsilon_{ij}u_j\lambda \sin\mu \frac{\partial \xi_2}{\partial T} + 2\epsilon_{ij}u_j \frac{\partial \xi_2}{\partial s}\right.$$

$$\left. + \epsilon_{ij}\lambda_j v \frac{\partial \xi_2}{\partial T}\right)$$

$$H_i = -\frac{2\lambda_j \epsilon_{ij}}{3\lambda \cos\mu}\left(\lambda \cos\mu \frac{\partial v}{\partial T} - U_t \frac{\partial \xi_2}{\partial T} + v \frac{\partial \xi_2}{\partial s}\right)$$

$$+ \frac{1}{\lambda \cos\mu}\left(\frac{u_i}{v}\lambda^2 \frac{\partial v}{\partial T} - \frac{u_i}{v}\lambda \sin\mu \frac{\partial v}{\partial s} - u_j \epsilon_{ij}\lambda^2 \frac{\partial \xi_2}{\partial T} + 2u_j \epsilon_{ij}\lambda \sin\mu \frac{\partial \xi_2}{\partial s}\right.$$

$$\left. + \lambda_j \epsilon_{ij}\lambda \cos\mu \frac{\partial v}{\partial T} - u_j \frac{\epsilon_{ij}}{v}\lambda \cos\mu \frac{\partial v}{\partial s} - \epsilon_{ij}\lambda_j v \sin\mu \frac{\partial \xi_2}{\partial T}\right)$$

$$I = -\frac{2v}{3}\left(\lambda \cos\mu \frac{\partial v}{\partial T} - U_t \frac{\partial \xi_2}{\partial T} + v \frac{\partial \xi_2}{\partial s}\right)$$

$$+ \frac{v}{\lambda \cos\mu}\left(\lambda^2 \cos^2\mu \frac{\partial v}{\partial T} + \lambda^2 \frac{\partial v}{\partial T} - \lambda \sin\mu \frac{\partial v}{\partial s} - \lambda^2 \sin\mu\, v \cos\mu \frac{\partial \xi_2}{\partial T}\right)$$

$$J = -\frac{v}{\lambda \cos\mu}\left(\lambda \sin\mu \frac{\partial v}{\partial T} - \frac{\partial v}{\partial s} - \lambda v \cos\mu \frac{\partial \xi_2}{\partial T}\right)$$

$$U_n = u_i \epsilon_{ij} \lambda_j$$

and

$$U_t = u_i \lambda_i$$

In problems involving the matching of conditions across discontinuities in non-viscous flow one uses the conservation equations around two types of contours, first around contours like PQRS, where PQRS contains a shock discontinuity

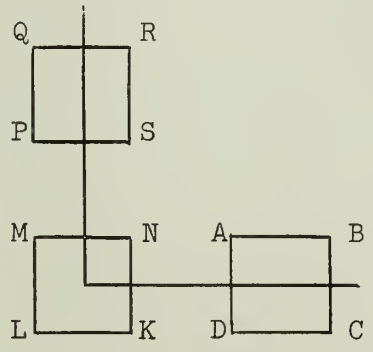

Figure 3.1

which is a discontinuity across which matter flows, and secondly around contours like ABCD, where ABCD contains a stream line discontinuity across which no matter flows. In each case one lets the region shrink to the line of discontinuity, and then by appealing to the fact that the result is true for arbitrary lengths of these discontinuities, or by differentiating with respect to the length, arrives at the conclusion that the jump in certain quantities is zero.

To simplify the viscous problem we assume that there will be regions where viscosity is important corresponding to the shocks and slip streams of non-viscous flow, but that elsewhere the viscous terms will be negligible. Thus we will consider the flow pattern shown in Figure 3.2 as the viscous analogy of the non-viscous pattern of Figure 3.1. We can shrink contours like PQRS and

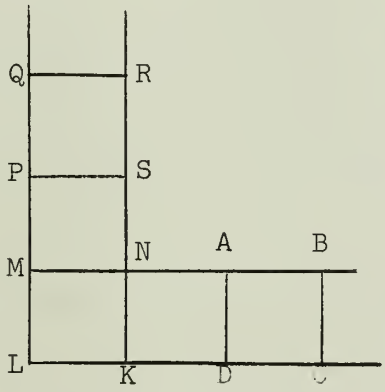

Figure 3.2

ABCD until PQ, RS, AB, and CD are the bounding lines of the adjacent non-viscous regions, and so have relations between the flows at PQ and RS and between the flows at AB and CD. KLMN will be the intersection region where two or more transition regions interact.

In the non-viscous problem the definition of the various lines like PS, QR, DA, BC, etc., presented no problem since as the limit was taken they shrank to zero and the result was independent of their choice. In the viscous case the complexity of the results may depend strongly on their choice.

The contour PQRS is analogous to a contour containing a shock discontinuity, so we will call a viscous transition region a shock region if matter flows in one boundary between the region and the adjacent non-viscous region and out of the other. Thus PQ and RS are boundaries between the viscous and non-viscous regions. We will define PS and QR as stream lines since these are independent of the choice of coordinates.

We will classify a region as a slip stream region if no matter flows across the boundaries between it and the adjacent non-viscous regions. Thus if ABCD is a slip stream region, AB and DC will be stream lines, or lines of constant s. To simplify the treatment we will define AD and BC as lines of constant T. Since we can expect a region of this type to originate at some disturbance such as the intersection region KLMN we can define T = 0 to be one boundary of KLMN, for example, KN. To define KN relative to the flow we must look for characteristics that distinguish the intersection region from the slip stream region. In this case we may expect the intersection region to have some of the characteristics of a shock region, namely, large derivatives of flow variables along the stream lines. We do not expect these in a slip stream region, so a definition of KN is the boundary between regions where the viscous forces due to terms such as $w\frac{\partial v}{\partial T}$ are significant or are not so.

One further simplifying assumption we will make is that the "shock thickness" is a constant along its length, or that the length of the stream lines within the shock region is constant. This means that if T = 0 is defined to be the boundary RSNK, then QPML is also a line of constant T.

The effect of the assumptions will be to make viscous effects due to terms in $w\frac{\partial v}{\partial T}$ dominant in a shock region, while viscous effects due to terms in $w\frac{\partial v}{\partial s}$ will dominate in a slip stream region. The intersection region will have characteristics of the adjacent shock and slip stream regions.

We have tacitly assumed continuity conditions across boundaries which we should explicitly enumerate. Along stream lines we will assume that all flow variables are continuous. Across stream lines we must have ξ_2 continuous as this is a geometric variable. Since we are taking account of viscous forces which arise from the derivatives of the velocity we must also consider the velocity to be continuous and differentiable (the derivates need not be continuous) so that across stream lines the velocity is continuous. If the derivatives of velocity have jumps across stream lines, the viscous forces will have jumps so that we may expect the pressure to have discontinuities also. The pressure discontinuities will arise at the boundaries of stream line regions because of the approximations to the flow. If we could solve the equations exactly they would not arise.

3.2 When we examine a shock region, supposed bounded by the two lines $T = T_0$ and $T = T_1$, we differentiate equations (3.1.1) - (3.1.3) with respect to s in a similar manner to the method of non-viscous theory. The resulting equations are

$$\left[\rho U_n \right]_{T_0}^{T_1} = 0 \qquad (3.2.1)$$

$$\left[\rho U_n u_i + p\lambda_j \epsilon_{ij} + wH_i \right]_{T_0}^{T_1} = \frac{\partial}{\partial s} \int_{T_0}^{T_1} \left[pu_j \frac{\epsilon_{ij}}{v} + wG_i \right] dt \qquad (3.2.2)$$

and

$$\left[\rho U_n \frac{v^2}{2} + \frac{\gamma}{\gamma - 1} pU_n + wI \right]_{T_0}^{T_1} = - \frac{\partial}{\partial s} w \int_{T_0}^{T_1} J dT \quad . \qquad (3.2.3)$$

The lines $T = T_0$ and T_1 can be considered as belonging to the non-viscous regions, so that the viscous terms on the left hand sides of (3.1.1) - (3.1.3) are zero; therefore, the remainder of the left hand sides are the set of R-H equations equated to terms of the right hand sides instead of to zero. Thus the right hand sides represent correction terms which must be estimated to the accuracy required.

To estimate these terms we must have some knowledge of the flow in the interior of the shock region. In order to understand the structure of the region we will examine a shock region where all functions are independent of s, that is, the region is one-dimensional. We will study a linear approximation to the flow in this region and then, in Section 3.3, extend this approximation to a shock region where derivatives with respect to s are small but not zero.

If the shock is one-dimensional the right hand sides vanish so we are left with the R-H equations alone. To the approximation that the flow is non-viscous outside of the shock region these are correct, but we have not yet determined the thickness of that region. To do that we must also consider equations (3.2.1) - (3.2.3) evaluated at various interior points of the shock region. Then we get the equations

$$\rho v = M = \rho_0 v_0 \tag{3.2.4}$$

$$Mv + p + \frac{4}{3} w \frac{dv}{dT} = Mv_0 + p_0 \tag{3.2.5}$$

$$M \frac{v^2}{2} + pv \frac{\gamma}{\gamma - 1} + \frac{4}{3} wv \frac{dv}{dT} = M \frac{v_0^2}{2} + p_0 v_0 \frac{\gamma}{\gamma - 1} \tag{3.2.6}$$

which have the solution

$$T = \frac{8w}{3M(\gamma+1)(v_0+v_1)} [v_0 \log(v_0 - v) - v_1 \log(v - v_1)] \tag{3.2.7}$$

$$\rho = \frac{M}{v}$$

$$p = Mv_0 - p_0 - Mv + M(\gamma + 1)\frac{(v_0 - v)(v - v_1)}{2v} \quad . \qquad (3.2.9)$$

The flow variables p, ρ and v attain their extreme values p_0, ρ_0 and v_0 or p_1, ρ_1 and v_1 at $T = -\infty$ or $+\infty$ respectively. To simplify the solution we could use (3.2.7) - (3.2.9) for part of the plane, say from $T = -\frac{\Delta}{2}$ to $T = \frac{\Delta}{2}$ and a non-viscous flow in the remainder of the plane. This would correspond to picking points where the flow had attained say 95% of its final value, or any other suitable value. Another common definition of shock thickness is to relate it to the total jump in a flow variable divided by the maximum value of the variable's derivative in the T direction. This corresponds to taking that variable to be linear and fitting it at the point of maximum slope. Both of these definitions lead to a shock thickness of the form $\frac{aw}{m_0^2 - 1}$ for weak shocks, where m_0 is the oncoming Mach number and a is a constant. This requires that a solution of the flow equations be known, as indeed one is for the one-dimensional shock. With the two-dimensional flow in mind we look for approximations to the flow in the one-dimensional problem that can be defined in terms of the flow equations (3.2.4) - (3.2.6) only. The simplest approximation we can make which takes account of viscosity is a linear one for p and v where for T lying between T_0 and $T_1 = T_0 + \Delta$ we write

$$p = p_0 + (p_1 - p_0)\frac{T - T_0}{\Delta}$$

$$v = v_0 + (v_1 - v_0)\frac{T - T_0}{\Delta}$$

$$\rho = \frac{M}{v} \quad .$$

These satisfy (3.2.4) so we must choose Δ to try to satisfy (3.2.5) and (3.2.6) in some manner. Since in this approximation (3.2.5) is always in error by

$\frac{4w}{3\Delta}(v_1-v_0)$ we can choose Δ to satisfy only (3.2.6) at one point, say the midpoint $T = T_0 + \Delta/2$. If a higher order approximation were to be used we could satisfy both equations at more points. Substituting $T = T_0 + \Delta/2$ into the approximations and evaluating (3.2.6) we get

$$\Delta = \frac{16w(v_0 + v_1)(\gamma - 1)}{3(\gamma - 1)\, M(v_0 - v_1)} \quad .$$

From (1.2.1) - (1.2.4)

$$v_0 - v_1 = \frac{2v_0}{\gamma + 1} \frac{m_0^2 - 1}{m_0^2}$$

so the thickness of the shock in this approximation is $O\left(\frac{w}{m_0^2 - 1}\right)$ for weak shocks where $m_0^2 \approx 1$.

3.3 In this section we will discuss the nature of an approximation to the structure of a two-dimensional shock region similar to that used for the one-dimensional shock region in the previous section. The approximations used inside the shock region are

$$\}_\nu = a_\nu + b_\nu(s - s_0) + c_\nu(T - T_0 + d_\nu(s - s_0(T - T_0) \qquad \nu = 1, 2, \ldots, 5$$

where $\}_1$, $\}_2$, $\}_3$, $\}_4$ and $\}_5$ are p, $\}_2$, v, λ and α respectively. Equation (3.2.1) specifies ρ in terms of these other variables. $T = T_0$ and $T = T_1$ are the two boundaries of the shock region. These forms for the variables will be assumed for T between T_0 and T_1 only. We have already assumed that the thickness of the shock is a constant. Let us assume that the equations have already been used to determine it in the form $\frac{aw}{m_0^2 - 1}$.

We will substitute the relations (3.3.1) into the conservation equations (3.2.2) and (3.2.3). In addition we will substitute them into the two geometric equations

$$\left[\lambda\right]_{T_0}^{T_1} = \int_{T_0}^{T_1} \cos\mu \, \frac{\partial \xi_2}{\partial s} \, dT \tag{3.3.2}$$

$$\left[\alpha\right]_{T_0}^{T_1} = \int_{T_0}^{T_1} \frac{\sin\mu}{\lambda} \, \frac{\partial \xi_2}{\partial s} \, dT \tag{3.3.3}$$

We are assuming that the flow ahead of the shock region is known and that the "shock angle" α is known on one boundary. We will take $\lambda = 1$ on this boundary. These conditions specify ten of the twenty unknowns of (3.3.1).

The five equations (3.2.2), (3.2.3), (3.3.2) and (3.3.3) are then functions of s which would vanish identically if the approximations (3.3.1) were exact solutions. We can pick the remaining ten unknowns of (3.3.1) to satisfy the five equations at $s = s_0$ and the derivatives of these five equations with respect to s at $s = s_0$. Thus the flow behind the shock region is specified in terms of the flow ahead of the region, the angle of the shock region boundary and its first derivative with respect to s. If we had taken a quadratic in s instead of a linear approximation, the flow behind the shock region would also have depended on the second derivative of the shock angle.

We are going to examine these equations in some detail when the flow ahead of the shock region is nearly sonic since this will simplify the equations and this is the case in which we are most interested as was mentioned in Section 3.1.

We will assume that the Mach number of the flow ahead of the shock region is given by

$$m_0^2 = 1 + \frac{1}{2}\epsilon^2$$

and that the angle of the boundary $T = T_1$ of the region is

$$\alpha = \frac{\pi}{2} + a\epsilon$$

where a is of order 1 in ϵ but may depend on ϵ. We will denote differentiation at the value of s under consideration with respect to a by ' and differentiation with respect to s by $\dot{}$. Thus we have

$$k = \dot{\alpha} = \dot{a}\epsilon \quad .$$

Let us take the flow ahead to be uniform with $\rho_0 = p_0 = 1$. The R-H equations have the solutions

$$\begin{aligned} \bar{U}_n &= \sqrt{\gamma}\left(1 - \frac{3-\gamma}{2(\gamma+1)}(1 - a^2)\epsilon^2\right) + O(\epsilon^4) \\ \bar{p} &= 1 + \frac{2\gamma}{\gamma+1}(1 - a^2)\epsilon^2 + O(\epsilon^4) \\ \bar{U}_t &= -\sqrt{\gamma}\, a\epsilon\left(1 + \frac{1}{6}(3 - a^2)\epsilon^2\right) + O(\epsilon^5) \end{aligned} \tag{3.3.4}$$

and the corresponding derivatives are

$$\begin{aligned} \dot{\bar{p}} &= -\frac{4\gamma}{\gamma+1}\, a\epsilon k + O(\epsilon^7) \\ \dot{\bar{U}}_n &= \sqrt{\gamma}\,\frac{3-\gamma}{\gamma+1}\, a\epsilon k + O(\epsilon^7) \\ \dot{\bar{U}}_t &= -\sqrt{\gamma}\, k - \frac{\epsilon^2}{2}\sqrt{\gamma}\,(1 - a^2)k + O(\epsilon^8) \end{aligned} \tag{3.3.5}$$

where k has been assumed to have order ϵ^4. The reason for this assumption is that it can be shown to be consistent with later work on two and three shock region intersections. In addition we will assume that the shock thickness Δ is of order ϵ^{-2} (this is in agreement with the results of one-dimensional flow) and that the solutions of the equations for the linear approximations are of the form

$$
\begin{aligned}
U_n &= \bar{U}_n - \frac{\sqrt{\gamma}}{\gamma+1}\left(\psi\epsilon^2 + \nabla\psi\epsilon^4\right) + O(\epsilon^6) \\
p &= \bar{p} + \frac{\gamma}{\gamma+1}\left(\phi\epsilon^2 + \nabla\phi\epsilon^4\right) + O(\epsilon^6) \\
U_t &= \bar{U}_t + \gamma\epsilon^3 + O(\epsilon^5) \\
\dot{p} &= \dot{\bar{p}} + \frac{\gamma}{\gamma+1} k(\phi'\epsilon + \nabla\phi'\epsilon^3) + O(\epsilon^9) \\
\dot{U}_n &= \dot{\bar{U}}_n - \frac{\sqrt{\gamma}}{\gamma+1} k(\psi'\epsilon + \nabla\psi\epsilon^3) + O(\epsilon^9) \\
\dot{U}_t &= \dot{\bar{U}}_t + \gamma' k\epsilon^2 + O(\epsilon^8)
\end{aligned}
\qquad (3.3.6)
$$

where ψ, ϕ, γ... etc. are independent of ϵ .

When we evaluate the integrals below we can use the trapezoidal rule, as errors will have orders in ϵ at least as high as those to be neglected in other terms evaluated. It may be verified that when equations (3.2.2) and (3.2.3) are used, the viscous terms wG_i and wJ will have a higher order in ϵ than any terms that are kept. It will be verified for the term $\frac{\partial}{\partial s}\int_{T_0}^{T_1} wJdT$ in Appendix I, the proof for other terms is similar.

We will need $\frac{\partial \xi_2}{\partial s}$ evaluated behind the shock region.

$$\xi_2 = \alpha + \mu = \frac{\pi}{2} + a\epsilon + \tan^{-1}\frac{U_t}{U_n} \quad .$$

Substituting from (3.3.6) and from (3.3.4) we get

$$
\begin{aligned}
\xi_2 &= \frac{\pi}{2} + a\epsilon - \tan^{-1}\left\{\frac{a\epsilon + \frac{1}{6}(3-a^2)\epsilon^3 a - \frac{\eta}{\sqrt{\gamma}}\epsilon^3}{1 - \frac{3-\gamma}{2(\gamma+1)}(1-a^2)\epsilon^2 - \frac{\psi}{\gamma+1}\epsilon^2}\right\} + O(\epsilon^5) \\
&= \frac{\pi}{2} + \frac{\eta}{\sqrt{\gamma}}\epsilon^3 - \frac{2}{\gamma+1}\epsilon^3 a(1-a^2) - \frac{\psi a}{\gamma+1}\epsilon^3 + O(\epsilon^5) \quad .
\end{aligned}
$$

Differentiating with respect to s we get

$$\frac{\partial \xi_2}{\partial s} = \gamma' \frac{k\epsilon^2}{\sqrt{\gamma}} - \frac{2}{\gamma+1} k\epsilon^2(1 - 3a^2) - \gamma \frac{\epsilon^2 k}{\gamma+1} - \gamma \cdot \frac{ak\epsilon^2}{\gamma+1} + O(\epsilon^8) . \tag{3.3.7}$$

We must evaluate λ and α ahead of the shock region. From (3.3.2) and (3.3.3) we have

$$1 - \lambda_{T_0} = \int_{T_0}^{T_1} \frac{\partial \xi_2}{\partial s} \cos(\xi_2 - \alpha) d\tau$$

$$= \frac{\Delta}{2} \left(\frac{\partial \xi_2}{\partial s} \right)_{T_1} + O(\epsilon^6) \tag{3.3.8}$$

and

$$\alpha_{T_1} - \alpha_{T_0} = \int_{T_0}^{T_1} \frac{\partial \xi_2}{\partial s} \sin(\xi_2 - \alpha) d\tau$$

$$= O(\epsilon^5) \quad . \tag{3.3.9}$$

We apply the conservation equations (3.2.2) which can be written as

$$\left[\rho U_n u_i + p\lambda_j \epsilon_{ij} \right]_{T_0}^{T_1} = \frac{\partial}{\partial s} \int_{T_0}^{T_1} p u_j \frac{\epsilon_{ij}}{v} d\tau + \delta_i \tag{3.3.10}$$

where the δ_i are the viscous terms of orders ϵ^6 and ϵ^5 for i = 1 and 2 respectively. The proof of a similar result for the energy equation is discussed in Appendix I. Multiplying (3.3.10) by λ_{iT_1} and summing on i we get

$$\left[\rho U_n \frac{U_t}{\lambda} \right]_{T_0}^{T_1} = \lambda_{iT_1} \frac{1}{2} \Delta \frac{\partial}{\partial s} \left(p U_j \frac{\epsilon_{ij}}{v} \right)_{T_1} + O(\epsilon^5),$$

on using the trapezoidal rule to evaluate the integral and using the fact that the flow ahead of the shock is independent of s.

The right hand side can be written as

$$\lambda_{iT_1} \frac{\Delta}{2} \left(\frac{U_i}{v} p \frac{\partial \xi_2}{\partial s} + u_j \frac{\epsilon_{ij}}{v} \frac{\partial p}{\partial s} \right) + O(\epsilon^5)$$

$$= \frac{\Delta}{2} \left(p \sin \mu \frac{\partial \xi_2}{\partial s} - \cos \mu \frac{\partial p}{\partial s} \right) + O(\epsilon^5)$$

since $\lambda_{T_1} = 1$. From (3.3.7) $\frac{\partial \xi_2}{\partial s} = O(\epsilon^7)$ and from (3.3.4) $\mu = \pi/2 + O(\epsilon)$. Therefore this can be written as $-\frac{\Delta}{2} \frac{\partial p}{\partial s} + O(\epsilon^5)$. From (3.2.1) the left hand side is

$$(\rho U_n)_{T_0} \left[\frac{U_t}{\lambda} \right]_{T_0}^{T_1} = (\rho U_n)_{T_0} \left[\frac{\bar{U}_t}{\lambda} \right]_{T_0}^{T_1} + (\rho U_n)_{T_0} \eta \epsilon^3$$

$$= \sqrt{\gamma} \eta \epsilon^3$$

by (3.3.4). Therefore

$$\eta \epsilon^3 = -\frac{\Delta}{2\sqrt{\gamma}} \frac{\partial p}{\partial s} + O(\epsilon^5) . \qquad (3.3.11)$$

Differentiating with respect to s we get $\eta' = 0$ since $\frac{\partial^2 p}{\partial s^2} = 0$ by the linearity assumption. Multiplying (3.3.10) by $\epsilon_{ij} \lambda_{jT_1}$ we get

$$\left[\rho U_n \frac{U_n}{\lambda} + p\lambda \right]_{T_0}^{T_1} = \epsilon_{ik} \lambda_{kT_1} \frac{\Delta}{2} \frac{\partial}{\partial s} \left(p u_j \frac{\epsilon_{ij}}{v} \right) + O(\epsilon^6)$$

or

$$\left[\rho \frac{U_n^2}{\lambda} + p \right]_{T_0}^{T_1} = \left[p(1 - \lambda) \right]_{T_0}^{T_1} + \epsilon_{ik} \lambda_{kT_1} \frac{\Delta}{2} \frac{\partial}{\partial s} \left(p u_j \frac{\epsilon_{ij}}{v} \right) + O(\epsilon^6) .$$

By use of (3.3.8) the right hand side can be written as

$$-\frac{\Delta}{2}\ \frac{\partial \xi_2}{\partial s}\ T_1 + \left(\sin\mu\ \frac{\partial p}{\partial s} + p\cos\mu\ \frac{\partial \xi_2}{\partial s}\right)_{T_1}\ \frac{\Delta}{2} + O(\epsilon^6)$$

$$= a\epsilon\dot{p}\ \frac{\Delta}{2} + O(\epsilon^6)\ .$$

On substituting from (3.3.6) and using (3.2.1) we get

$$-\frac{\gamma}{\gamma+1}(\psi + \nabla\psi\epsilon^2)\epsilon^2 + \frac{\gamma}{\gamma+1}(\phi + \nabla\phi\epsilon^2)\epsilon^2 + \frac{1}{2}\frac{\gamma}{\gamma+1}\psi(1-a^2)\epsilon^4$$

$$= -a\epsilon\ \frac{\partial p}{\partial s}\ \frac{\Delta}{2} + O(\epsilon^6)\ . \qquad (3.3.12)$$

The energy equation (3.2.3) may be written as

$$\int_{T_0}^{T_1}\left[M\frac{v^2}{2} + \gamma p\frac{U_n}{\gamma+1}\right] = O(\epsilon^6)$$

since the viscous terms have orders ϵ^6 and higher, as is verified in Appendix I. Using $\lambda^2 v^2 = U_n^2 + U_t^2$ and (3.3.8) we may rewrite this as

$$\int_{T_0}^{T_1}\left[M\ \frac{U_n^2 + U_t^2}{2\lambda^2} + \frac{\gamma}{\gamma-1}\ p\ \frac{U_n}{\lambda}\right] = \int_{T_0}^{T_1}\left[\frac{\gamma}{\gamma-1}\ pU_n\left(\frac{1}{\lambda} - 1\right)\right] + O(\epsilon^6)$$

$$= -\frac{\gamma\sqrt{\gamma}}{\gamma-1}\ \frac{\Delta}{2}\ \frac{\partial \xi_2}{\partial s} + O(\epsilon^6)\ .$$

Substituting from (3.3.6) we obtain

$$\frac{\gamma^2\sqrt{\gamma}}{\gamma^2-1}\left\{\phi+\nabla\phi\epsilon^2-\psi-\nabla\psi\epsilon^2\right\}\epsilon^2+\frac{\gamma\sqrt{\gamma}}{2(\gamma+1)^2}\psi^2\epsilon^4-\frac{\gamma^2\sqrt{\gamma}}{(\gamma-1)^2(\gamma-1)}\psi\phi\epsilon^4$$

$$-\gamma a\epsilon^4\eta-\frac{\gamma\sqrt{\gamma}\psi(1-a^2)}{(\gamma+1)^2(\gamma-1)}(\gamma^2-\gamma+1)\epsilon^4-\frac{\gamma^2\sqrt{\gamma}}{(\gamma-1)^2(\gamma-1)}\frac{3-\gamma}{2}(1-a^2)\phi\epsilon^4$$

$$=\frac{\gamma\sqrt{\gamma}}{\gamma-1}\frac{\Delta}{2}\frac{\partial\zeta_2}{\partial s}+O(\epsilon^6)\quad . \tag{3.3.13}$$

From both (3.3.12) and (3.3.13) we find by equating terms in ϵ^2 that $\phi=\psi$. This is valid only at the value of s we are considering. It is not a functional relation because we are unable to satisfy the equations identically in s with this approximation. In addition we insist that the derivatives with respect to s of (3.3.12) and (3.3.13) are satisfied for this value of s. It can be seen immediately that they will lead to relations $\phi'=\psi'$ at this value of s. Multiplying (3.3.13) by $\frac{\gamma-1}{\gamma\sqrt{\gamma}}$, subtracting (3.3.12) and substituting for η from (3.3.11) we get

$$-\frac{\psi^2\epsilon^4}{2(\gamma+1)}-\frac{a\Delta\epsilon}{2\gamma}\frac{\partial p}{\partial s}-\psi\frac{(1-a^2)}{\gamma+1}\epsilon^4+\frac{\Delta}{2}\frac{\partial\zeta_2}{\partial s}=0 \tag{3.3.14}$$

Substituting for $\frac{\partial\zeta_2}{\partial s}$ from (3.3.7) and for $\frac{\partial p}{\partial s}$ from (3.3.6) we get

$$(4a-\phi')La=\frac{1}{2}(L+\psi)(\psi+2(1-a^2))+O(\epsilon^2) \tag{3.3.15}$$

where $L=\epsilon^{-2}k\Delta$. We must also equate the differential coefficients of (3.3.12) and (3.3.13) to 0 at the value of s under consideration to get two further relations between ψ, ψ', ϕ and ϕ'. We have already remarked that this will lead to $\phi'=\psi'$. Since (3.3.14) is a linear combination of (3.3.12) and (3.3.13) we can differentiate this. Noting that $\frac{\partial^2\zeta_2}{\partial s^2}=\frac{\partial^2 p}{\partial s^2}=0$ by the linear approximation we get

$$-\psi\phi'-(1-a^2)\phi'+2a\psi+\frac{L}{2}(4a-\phi')=O(\epsilon^2)$$

or

$$(4a - \phi')(1 - a^2 + \psi + \frac{L}{2}) = (2(1 - a^2) + \psi)2a + O(\epsilon^2). \qquad (3.3.16)$$

One solution of (3.3.15) and (3.3.16) to order 1 in ϵ is

$$\phi' = 4a, \qquad \psi = -2(1 - a^2) .$$

This corresponds to $p = 1$, $U_n = U_{n0}$, or in other words there is no change through the shock region. The non-trivial solutions are

$$\psi = \frac{-3L}{4} - \frac{1 - a^2}{2} \pm \sqrt{\left(\frac{1 + a^2}{2} + \frac{3L}{4}\right)^2 + 4La^2} \Big\} . \qquad (3.3.17)$$

If $k = 0$ then $L = 0$. The solutions are then $\psi = 0$ or $-(1 - a^2)$ and $\phi' = 0$ or ∞ respectively. Since the latter solution would involve infinite derivatives of velocity outside of the viscous region, it will be discarded. When L is non-zero it is not obvious which branch of the solution should be used. We will use the branch corresponding to the zero solution at $L = 0$.

The flow variables can now be expressed by equations (3.3.17) and

$$\phi' = \frac{-1}{2La}[\psi + 2(1 - a^2)][\psi + L] + 4a \qquad (3.3.18)$$

$$p = 1 + \frac{\gamma}{\gamma + 1}[2(1 - a^2) + \psi]\epsilon^2 + O(\epsilon^4) \qquad (3.3.19)$$

$$c^2 = \gamma(1 + \frac{\gamma - 1}{\gamma + 1}[2(1 - a^2) + \psi]\epsilon^2) + O(\epsilon^4) \qquad (3.3.20)$$

$$v^2 = \gamma(1 - \frac{2}{\gamma + 1}[2(1 - a^2) + \psi]\epsilon^2) + O(\epsilon^4) \qquad (3.3.21)$$

$$-2 = \frac{\pi}{2} - \frac{2}{\gamma + 1}a(1 - a^2)\epsilon^3 - \frac{a\psi\epsilon^3}{\gamma + 1} - \frac{L\epsilon^3}{2(\gamma + 1)}(4a - \phi') + O(\epsilon^5) \qquad (3.3.22)$$

$$\frac{\partial p}{\partial s} = \frac{\gamma}{\gamma + 1} k\epsilon(\phi' - 4a) + O(\epsilon^7) \qquad (3.3.23)$$

$$\frac{\partial \zeta_2}{\partial s} = -\frac{2}{\gamma+1}\, k(1-3a^2)\epsilon^2 - \frac{k\epsilon^2}{\gamma+1}\psi - a\frac{k\epsilon^2}{\gamma+1}\phi' + O(\epsilon^8) . \qquad (3.3.24)$$

These will be referred to as the modified R-H equations. The correction terms depend only on L which is proportional to kw. We will assume that it is sufficient to introduce these extra terms only, and study the effects of these modifications on problems which have no solutions when the unmodified R-H equations are used.

3.4

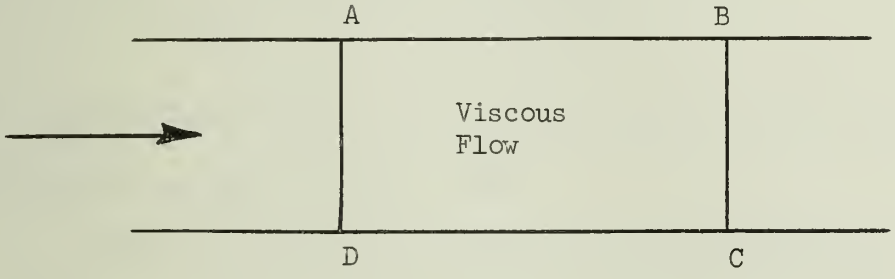

Figure 3.3

We will next study the viscous analogue of a slip stream. In non-viscous flow we applied the four conservation equations to a region ABCD containing a discontinuity. When the region was shrunk to the line of discontinuity we obtained equations that had a solution where the line of discontinuity was stream line, so that there was no mass flow across the discontinuity, and that $[pu_j \frac{\epsilon_{ij}}{v}] = 0$ across the stream line. This implied that ζ_1 and ζ_2 were continuous across the stream line.

In viscous flow we define the region between AB and DC (Figure 3.3) as a slip stream region if the boundaries AB and DC are stream lines and if the flow above AB and below DC is non-viscous to the approximation used. Across AB and DC we assume that ζ_2 and v are continuous. Continuity of ζ_1 is not assumed as was stated in Section 3.1. ρ is defined to satisfy the conservation

of mass. We apply the conservation of momentum and energy, equations (3.1.2) and (3.1.3), where $s = s_0$ is CD and $s = s_1$ is AB and then differentiate with respect to T to get

$$\left[\int_{s_0}^{s_1} p u_j \frac{\epsilon_{ij}}{v} \right] = \frac{\partial}{\partial T} \int_{s_0}^{s_1} (\rho U_n u_i + p\lambda_j \epsilon_{ij} + wH_i) ds \qquad (3.4.1)$$

and

$$0 = \frac{\partial}{\partial T} \int_{s_0}^{s_1} (\rho U_n \frac{v^2}{2} + \frac{\gamma}{\gamma - 1} pU_n + wI) ds \qquad (3.4.2)$$

where the left hand sides are evaluated in the non-viscous region so that the viscous terms are zero on these sides.

Using the linear approximations (3.3.1) for each variable we get a set of relations involving the coefficients of (3.3.1) which could be satisfied identically in T if the approximations were exact solutions. We will enforce the equations at a given $T = T_1$ and also ask that the T-derivatives of (3.4.1) vanish at $T = T_1$. If the viscous terms are small the integrand in (3.4.2) is approximately the enthalpy, a quantity which is nearly constant along a stream line. We want to enforce this over as many stream lines in the viscous region as possible. With a linear approximation we can only enforce it on two stream lines which we will take as $s = s_0$ and $s = s_1$, so we have

$$\frac{\partial}{\partial T} (\rho U_n \frac{v^2}{2} + \frac{\gamma}{\gamma - 1} pU_n + wI) = 0 \qquad \text{at } s = s_0 \text{ and } s_1 \qquad (3.4.3)$$

Equations (3.3.1) and its T derivatives express $\left[\int_{s_0}^{s_1} p u_j \frac{\epsilon_{ij}}{v} \right]$ and $\frac{\partial}{\partial T} \left[\int_{s_0}^{s_1} p \epsilon_{ij} \frac{U_j}{v} \right]$ in terms of the flow in the interior of the slip stream region. If the flow along AD is known then we can use (1.2.9) and (1.2.10) to

find $\frac{\partial \lambda}{\partial T}$ and $\frac{\partial \alpha}{\partial T}$ at A and D. (3.4.3) expresses $\frac{\partial p}{\partial T}$ inside the slip stream region in terms of the other variables. The continuity conditions give ζ_2 and v inside the slip stream identical to ζ_2 and v in the non-viscous regions at the boundaries AB and DC, so that (3.4.1) will give a set of four relations between the flow either side of the slip stream region and p, λ and α along AD. We are going to examine these relations in some detail for a "weak" slip stream region, which is a slip stream region with the following properties: If the flow variables are scaled so that at some point in the flow $m = p = \rho = 1$ and $\zeta_2 = \pi/2$ then throughout the region we have

$$p, \rho, m \frac{c^2}{\gamma} \text{ and } \frac{v^2}{\gamma} \text{ have the form } 1 + O(\epsilon^2)$$

$$\zeta_2 = \frac{\pi}{2} + O(\epsilon^3)$$

$$\alpha = \frac{\pi}{2} + O(\epsilon)$$

$$\lambda = 1 + O(\epsilon^4)$$

or in other words they have the same magnitude as the variables behind a weak shock region. In addition we will assume that the T-derivatives of the variables along AB and DC in the non-viscous region have the same magnitude as the T-derivatives behind a weak shock region, namely

$$\frac{\partial \zeta_1}{\partial T} = O(\epsilon^4), \quad \frac{\partial \zeta_2}{\partial T} = O(\epsilon^5),$$

that the thickness of the slip stream region $D = s_1 - s_0 = O(\epsilon^{-3})$ and that the pressure discontinuities at $s = s_0$ and $s = s_1$ are of order ϵ^4. These assumptions will be shown to be consistent for regular and Mach reflection later. We define $\lambda = 1$ at $T = T_1$ or AD, and since α is a linear function on AB, we define

$$\alpha = \alpha_0 + (\alpha_1 - \alpha_0)\frac{s - s_0}{D} \quad \text{on AD.}$$

Let

$$\alpha_i = \frac{\pi}{2} + b_i\epsilon + O(\epsilon^3) \qquad i = 0, 1.$$

Substituting into (3.4.1) we get

$$\left[pu_j \frac{\epsilon_{ij}}{v}\right]_{s_0}^{s_1} = \frac{\partial}{\partial T}\int_{s_0}^{s_1} \left(\rho U_n \frac{u_i}{v}\frac{\partial v}{\partial T} - \rho U_n u_j \epsilon_{ij}\frac{\partial \xi_2}{\partial T} + p\frac{u_i}{v}\frac{\partial \xi_2}{\partial s} - \lambda_j \epsilon_{ij}\frac{\partial p}{\partial T}\right) ds + \delta_i$$

where the δ_i are the viscous terms and have orders ϵ^5 and ϵ^4 for $i = 1$ and 2 respectively. A proof similar to that in Appendix I will verify this. Also we can verify that the viscous terms in (3.4.3) can be neglected. Substituting for $\frac{\partial p}{\partial T}$ from (3.4.3), neglecting terms of order δ_i and using $\frac{\partial v}{\partial T} = -\frac{c}{m}\frac{\partial \xi_1}{\partial T}$ from (1.2.11) on the non-viscous boundary we get for $i = 1$

$$\left[\xi_2\right]_{s_0}^{s_1} = -\frac{D}{2}\left\{b_1\left(\frac{\partial \xi_2}{\partial T}\right)_{s_1} + b_0\left(\frac{\partial \xi_2}{\partial T}\right)_{s_0}\right\}\epsilon + \frac{D}{2}\left\{\left(\frac{\partial \xi_1}{\partial T}\right)_{s_1}\left[\frac{\gamma+1}{\gamma}(p-1) - \epsilon^2\right]_{s_1} + \left(\frac{\partial \xi_1}{\partial T}\right)_{s_0}\left[\frac{\gamma+1}{\gamma}(p-1) - \epsilon^2\right]_{s_0}\right\} + O(\epsilon^5) \qquad (3.4.4)$$

and for $i = 2$

$$-\left[p\right]_{s_0}^{s_1} = \frac{D}{2}\epsilon\left\{b_1\left(\frac{\partial \xi_1}{\partial T}\right)_{s_1} + b_0\left(\frac{\partial \xi_1}{\partial T}\right)_{s_0}\right\} + \frac{D}{2}\gamma\left\{\left(\frac{\partial \xi_2}{\partial T}\right)_{s_1} + \left(\frac{\partial \xi_2}{\partial T}\right)_{s_0}\right\} + O(\epsilon^4)$$

where $\left(\frac{\partial \xi_i}{\partial T}\right)_{s_1}$ is evaluated in the non-viscous region on AB

and $\left(\frac{\partial \xi_i}{\partial T}\right)_{s_0}$ is evaluated in the non-viscous region on DC.

Differentiating (3.4.1) with respect to T and substituting we get

$$\frac{\partial}{\partial T}\left[\int_{s_0}^{s_1} pu_j \frac{\epsilon_{ij}}{v}\right] = \epsilon_i$$

where the ϵ_i represent the terms on the right hand sides of the T-derivatives of (3.4.1) and have orders ϵ^7 and ϵ^6 respectively. Thus we have

$$\left[\frac{\partial \xi_2}{\partial T}\right]_{s_0}^{s_1} = O(\epsilon^7) \tag{3.4.6}$$

and

$$\left[\frac{\partial \xi_1}{\partial T}\right]_{s_0}^{s_1} = O(\epsilon^6) \tag{3.4.7}$$

for i = 1 and 2 respectively. Equations (3.4.4) - (3.4.7) are four relations between p and ξ_2 on each boundary of the slip stream region and the flow along $T = T_1$ or AD. The only parameters on this boundary are b_0, b_1 and D, since v, ξ_2, p, λ and ρ are known as linear approximations from the continuity conditions and the additional assumptions. These assumptions were that the variables had certain orders which will be shown to be consistent. If the stream lines are straight and the pressure is constant along them, then the right hand sides of (3.4.4) and (3.4.5) are zero. The left hand sides then give

$$\left[p\right]_{s_0}^{s_1} = \left[\xi_2\right]_{s_0}^{s_1} = 0 ,$$

which are the matching conditions across a non-viscous stream line.

3.5 We have discussed linear approximations to shock regions and slip stream regions. Now we will consider two problems in which shock regions interact and produce a slip stream region. Such interactions occur in regular and Mach reflection of shocks shown in Figure 3.4. They also occur in other instances. In Figures 3.4 the region ARMB will be referred to as the intersection region. The line RM will correspond to the line AD of the previous section. For convenience ARMB is taken as the rectangle bounded by $s = s_0$ and s_1 and $T = T_0$ and T_1 in the s-T plane. In both cases the incident shock

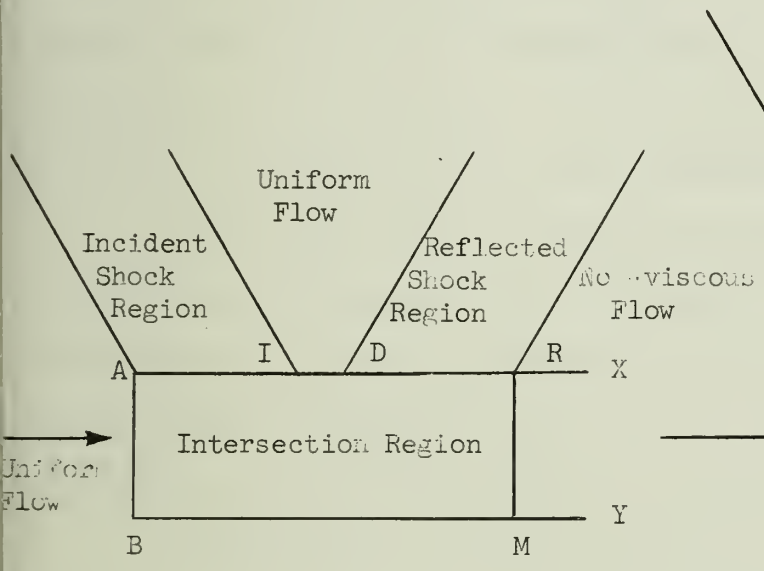

Figure 3.4a
Regular Reflection

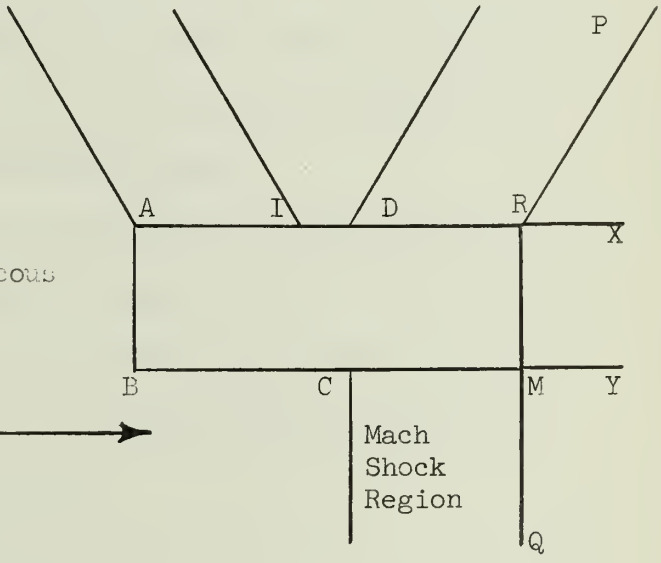

Figure 3.4b
Mach Reflection

is assumed to be known and to be straight so that the unmodified R-H equations (1.2.1) - (1.2.4) determine the flow in the region between the incident and reflected shock regions. RMYX is the slip stream region. AIDRX and BCMY are stream lines. We will define the thickness of the shock regions as

$\triangle_I$ = length along the stream line AI

$\triangle_R$ = length along the stream line DR

Δ_M = length along the stream line CM .

In addition let us define the lengths along the arcs RM and BM as D and Δ respectively. RM is defined as the boundary of the intersection region such that to the right of RM viscous forces such as $w\frac{\partial v}{\partial T}$ can be ignored as they were ignored in Section 3.4. We will define λ as 1 along PRMQ and take this line as $T = T_1$.

First the three-shock problem will be discussed and then the modifications for the two-shock problem indicated. Using the linear approximation we can calculate by the techniques of Section 3.3 the flow behind the reflected and Mach shocks from the flow ahead of each and the shock angles and curvatures α_R, α_M, k_R and k_M respectively. We will also use a linear approximation of the form (3.3.1) for the flow in the intersection region.

The method will be to determine the flow along RM by application of the conservation equations to the intersection region. Then the flow is known above and below the slip stream region and along the boundary RM, so that the techniques of Section 3.4 will give four relations between these flows from which α_R, α_M, k_R and k_M are to be evaluated.

Equations (3.3.2) and (3.3.3) applied at $s = s_0$ and $s = s_1$ will express α and λ at A and B in the intersection region in terms of ξ_2 throughout the intersection region and α_0 and α_1; ξ_2 is known at the four corners of the intersection region in terms of α_R, α_M, k_R and k_M; therefore its linear approximation is known in ARMB so that α and λ are determined in ARMB in terms of these variables and α_0 and α_1. The energy equation (3.2.3) may be applied at $s = s_0$ and $s = s_1$ to give p at R and M inside the intersection region in terms of these variables. Then the momentum equations (3.1.2) must be applied twice around suitable contours to give four equations that will determine Δ, D, α_0 and α_1 in terms of α_R, α_M, k_R and k_M. In the non-viscous

regions the flow along RX and MY can be expressed in terms of the flow along RP and MQ by equations (1.2.11), and hence in terms of α_R, α_M, k_R and k_M. Finally the four conditions across the slip stream region will give four equations in α_R, α_M, k_R and k_M.

We will derive the equations for a "weak" intersection region, which is such that the flow variables behave as indicated for a weak slip stream region. From (3.3.2) and (3.3.3) and the assumptions of the orders of the variables we have

$$\lambda_A = \lambda_B = 1 - \frac{\Delta}{2D} \int_{s_0}^{s_1} \left[\xi_2 \right]_{T_1} + O(\epsilon^6) \tag{3.5.1}$$

$$\begin{aligned} \alpha_A &= \alpha_1 + O(\epsilon^5) \\ \alpha_B &= \alpha_0 + O(\epsilon^5) \quad . \end{aligned} \tag{3.5.2}$$

We apply the conservation of momentum equations (3.1.2) separately around two halves of the intersection region divided by the stream line $s = \frac{s_0 + s_1}{2}$. To take account of the flow above $s = s_1$ and below $s = s_0$ we will evaluate the integrals along these lines in the regions adjacent to the intersection region, namely the shock and non-viscous regions. The viscous terms can once again be neglected. We get by using the trapezoidal rule for integration around the upper half of ARMB

$$\begin{aligned} \int_{T_0}^{T_1} \left[\frac{D}{8} \int_{s_0}^{3s_1} \left\{ \rho U_n u_i + p\lambda_j \epsilon_{ij} \right\} \right] &= \frac{\Delta}{2} I \left\{ \int_I^A Y_i \right\} + (\Delta - \Delta_I - \Delta_R)\left(\int Y_i \right)_I \\ &+ \frac{\Delta}{2} R \left\{ \int_I^R Y_i \right\} - \frac{\Delta}{2} \left(\left\{ \int_{T_0}^{T_1} Y_i \right\} \right)_{\frac{s_0+s_1}{2}} \\ &+ O(\epsilon^{n_i}) \end{aligned} \tag{3.5.3}$$

where $n_1 = 3$ and $n_2 = 2$, $\overset{3B}{\underset{A}{\{ X \}}}$ means $X_A + 3X_B$, and $Y_i = \rho u_j \frac{\epsilon_{ij}}{v}$; and around the lower half of ARMB

$$\int_{T_o}^{T_1}\left[\frac{D}{8}\int_{3s_o}^{s_1}\left\{\rho U_n u_i + p\lambda_j \epsilon_{ij}\right\}\right] = \frac{\Delta}{2}\int_{T_o}^{T_1}\left\{Y_i\right\}_{\frac{s_o+s_1}{2}} - (\Delta - \Delta_M)(Y_i)_B$$

$$- \frac{\Delta_M}{2}\overset{M}{\underset{C}{\{Y_i\}}} + o(\epsilon^{n_i}) . \qquad (3.5.4)$$

Taking three times the (3.5.3) minus (3.5.4) we get

$$\int_{T_o}^{T_1}\left[D(\rho U_n u_i + p\lambda_j \epsilon_{ij})_{s_1}\right] = \Delta\left[3(Y_i)_I - (Y_i)_A - \overset{R}{\underset{M}{\{Y_i\}}}\right] + \frac{3\Delta_R}{2}\overset{R}{\underset{I}{[Y_i]}}$$

$$- \frac{3\Delta_I}{2}\overset{I}{\underset{A}{[Y_i]}} + \frac{\Delta_M}{2}\overset{M}{\underset{C}{[Y_i]}} + o(\epsilon^{n_i}) \qquad (3.5.5)$$

and taking three times (3.5.4) minus (3.5.3) we get

$$\int_{T_o}^{T_1}\left[D(\rho U_n u_i + p\lambda_j \epsilon_{ij})_{s_o}\right] = \Delta\left[\overset{R}{\underset{M}{\{Y_i\}}} - \overset{A}{\underset{I}{\{Y_i\}}}\right] - \frac{\Delta_R}{2}\overset{R}{\underset{I}{[Y_i]}} - \frac{3\Delta_M}{2}\overset{M}{\underset{A}{[Y_i]}}$$

$$+ \frac{\Delta_I}{2}\overset{I}{\underset{A}{[Y_i]}} + o(\epsilon^{n_i}) \qquad (3.5.6)$$

We now evaluate the terms on the left hand sides to get the four equations

$$2D[\gamma(\xi_{2R} - \xi_{2A}) + (p_R - p_A)\epsilon b_1] + 2\Delta[3p_I - p_A - p_R - p_M]$$

$$= -3\Delta_R(p_R - p_I) + 3\Delta_I(p_I - p_A) - \Delta_M(p_M - p_A) + o(\epsilon^2) \qquad (3.5.7)$$

$$2D[\gamma(\xi_{2M} - \xi_{2A}) + (p_M - p_A)\epsilon b_0] + 2\Delta(p_R + p_M - p_I - p_A)$$

$$= \Delta_R(p_R - p_I) - \Delta_I(p_I - p_A) + 3\Delta_M(p_M - p_A) + O(\epsilon^2) \qquad (3.5.8)$$

$$\frac{\gamma+1}{\gamma} D(p_R - p_A)\left(\frac{p_R - p_A}{\gamma} - \frac{2\epsilon^2}{\gamma+1}\right) + 2\Delta(\xi_{2A} + \xi_{2R} + \xi_{2M} - 3\xi_{2I})$$

$$= 3\Delta_R(\xi_{2R} - \xi_{2I}) - 3\Delta_I(\xi_{2I} - \xi_{2A}) + \Delta_M(\xi_{2M} - \xi_{2A}) + O(\epsilon^3)$$

(3.5.9)

and

$$\frac{\gamma+1}{\gamma} D(p_M - p_A)\left(\frac{p_M - p_A}{\gamma} - \frac{2\epsilon^2}{\gamma+1}\right) + 2\Delta(\xi_{2I} + \xi_{2A} - \xi_{2R} - \xi_{2M})$$

$$= -\Delta_R(\xi_{2R} - \xi_{2I}) + \Delta_I(\xi_{2I} - \xi_{2A}) - 3\Delta_M(\xi_{2M} - \xi_{2A}) + O(\epsilon^3)$$

(3.5.10)

We have used the assumption that the pressures p evaluated at the same point in either the slip stream region or the non-viscous region agree to terms in ϵ^4. This can be verified from the energy equation (3.2.3) which implies that

$$\int_{T_0}^{T_1}\left[\frac{\rho v^3}{2} + pv\frac{\gamma}{\gamma-1}\right] = O(\epsilon^4) \qquad (3.5.11)$$

for the stream line $s = s_0$ or s_1 considered as belonging to either the intersection region or the shock and non-viscous regions. The conservation of mass

$$0 = \int_{T_0}^{T_1}\left[\rho U_n\right] = \int_{T_0}^{T_1}\left[\rho v\right] + O(\epsilon^4) \qquad (3.5.12)$$

is also satisfied on $s = s_0$ or s_1 in both regions. Since ρ, v and p at $T = T_0$ are independent of the region of evaluation for $s = s_0$ or s_1 and the

same is true for v at $T = T_1$, (3.5.11) and (3.5.12) imply that p and ρ are independent of the region of evaluation at R or M except for terms of $O(\epsilon^4)$ and higher.

3.6 In this section we will apply the modified and unmodified R-H equations and the modified matching conditions to determine the shock region pattern.

Let the oncoming flow be scaled so that

$$p_0 = \rho_0 = 1$$

$$m_0 = 1 + \frac{1}{2}\epsilon^2 \quad ;$$

assume that the angle of the incident shock is

$$\alpha = \frac{\pi}{2} + a\epsilon$$

and its curvature zero. The flow behind the incident shock region is given by the R-H equations (1.2.1) - (1.2.4). The Mach number behind the shock is

$$m_I^2 = 1 + (2a^2 - 1)\epsilon^2$$

and the direction of the flow is

$$\xi_{2I} - \frac{\pi}{2} = -\frac{2}{\gamma + 1}\, a(1 - a^2)\epsilon^3$$

radians to the horizontal. Let us define the reflected shock angle by

$$\alpha_R = \frac{\pi}{2} + \delta\epsilon_R - \frac{2a}{\gamma + 1}(1 - a^2)\epsilon^3$$

where $\epsilon_R^2 = (2a^2 - 1)\epsilon^2$ and the Mach shock angle by $\alpha_M = \pi/2 + \beta\epsilon$. Then the modified R-H equations (3.3.17) - (3.3.24) define the flow behind the shock regions in terms of β, k_M, δ and k_R with suitable substitutions. In the non-viscous regions we can write

$$\frac{\partial \xi_i}{\partial T} = A_{ij} \frac{\partial \xi_j}{\partial s}$$

from (1.2.11). Let $a_R = \delta$, $a_M = \beta$, $\epsilon_M = \epsilon$; then for $Z = R$ or M we have

$$\left(\frac{\partial \xi_1}{\partial T} \right)_Z = \frac{k_Z}{\gamma + 1} \frac{10a_Z^2 - 2 - \psi_Z + 2a_Z \phi_Z'}{1 - a_Z^2 + \psi_Z} + O(\epsilon^6) \qquad (3.6.1)$$

$$\left(\frac{\partial \xi_2}{\partial T} \right)_Z = - \frac{k_Z}{\gamma + 1} \epsilon_Z \frac{14a_Z^3 - 6a_Z - 5a_Z\psi_Z - \phi_Z'(3a_Z^2 - 1 - \psi_Z)}{1 - a_Z^2 + \psi_Z} + O(\epsilon^7). \qquad (3.6.2)$$

Equations (3.5.9) and (3.5.10) are linear equations in D and Δ, and thus express D and Δ in terms of β, δ, k_R and k_M. Substituting these relations into (3.5.7) and (3.5.8) we get b_0 and b_1 in terms of these variables. Substituting these, (3.6.1) and (3.6.2) into (3.4.4) and (3.4.5) we get two relations, and substituting (3.6.1) and (3.6.2) into (3.4.6) and (3.4.7) we get two more relations in β, δ, k_R and k_M. Thus we have four equations to solve for these four variables. One solution that can be seen immediately is $k_R = k_M = 0$, β and δ given by the solution of the non-viscous equations. This is obvious since $k_R = k_M = 0$ implies that derivatives with respect to T are 0 behind the shocks, and hence (3.4.6) and (3.4.7) are satisfied, leaving (3.4.4) and (3.4.5) for which the right hand sides vanish so that they give

$$\int_{s_0}^{s_0} [p] = \int_{s_0}^{s_1} [\xi_2] = 0$$

which are the usual non-viscous conditions. Solutions, if they exist, for the variables β, δ, k_R, k_M, D, Δ, b_0 and b_1 have the order that was assumed, so that the assumptions are consistent. The thicknesses of the shocks Δ_R, Δ_M and Δ_I are proportional to w, so from equations (3.5.7) - (3.5.10) we have that Δ

and D are proportional to w while b_0 and b_1 are independent of it. Then from (3.4.4) - (3.4.7) we find that k_R and k_M are inversely proportional to w but β and δ are independent of it. If a solution of these equations exists with either k_R or k_M non-zero, then as the viscosity w tends to zero, the non-zero k's will tend to ∞. Therefore we cannot have a singularity of this type in which the non-viscous R-H equations and matching conditions break down unless k is infinite when the viscosity is zero.

3.7 For regular reflection in viscous flow, there are two cases; reflection at a bounding wall and reflection at an axis of symmetry. In non-viscous flow these cases are identical. In reflection at a bounding wall in viscous flow the velocity is prescribed as zero on the wall. Since the fluid away from the wall is moving there will be a boundary layer acrosss which the velocity is non-linear (see Goldstein[5]). The reason for the non-linearity is that at the wall the $w \frac{\partial v}{\partial s}$ must be large so that viscous drag can slow the fluid while at the non-viscous boundary of the boundary layer, $w \frac{\partial v}{\partial s}$ must be small. Therefore, our linear approximation to v is inappropriate to reflection at a bounding wall. We will only consider reflection at a line of symmetry for which the only condition is that $\xi_2 = \pi/2$ along the stream line MY.

We replace the variables β and k_R by the velocity and its T derivative at M and write $\left(\frac{\partial \xi_2}{\partial T}\right)_M = \xi_{2M} - \pi/2 = 0$. Using these in the three shock equations and writing $\Delta_M = \Delta$ we will get a similar set of equations appropriate to regular reflection. All remarks concerning the behavior of δ, k_R, Δ, D, b_0 and b_1 in the three shock problem are applicable to these variables in the two shock problem. Since Smith[9] has observed regular reflection for weak shocks at incident angles for which no solution of the R-H equations exists, we hope that infinite curvature of the shock will allow solutions to these equations to exist when the viscosity tends to zero.

IV CONCLUSION

In Chapter II we studied the problem of a wedge in an oncoming supersonic non-viscous flow such that the shock was attached to the wedge. This was similar to the problem of regular reflection. We saw how by the method of characteristics we could solve for the flow behind the shock when the flow ahead was uniform, the wedge wall straight and the flow initially supersonic. Similar methods will suffice to treat the problem when the flow ahead is non-uniform and the wedge wall curved. When the flow emerging from the shock was initially subsonic an analytic solution existed only for a countable number of wedge angles. For other angles a class of singular solutions of the differential equations was found that satisfied the local boundary conditions, and this class was seen to have its analogy in pseudo-stationary flow. One problem that was not discussed was the use of other boundary conditions to determine the parameters of the solution. Until this has been treated, no discussion of uniqueness is possible.

The solution had a branch point singularity at the origin, similar to the singularity used by Guderly[6] (Chapter VI) in his discussion of an irrotational flow behind a shock. A branch point is also found in two-dimensional potential problems at an angle in an equi-potential line. The similarity is due to the fact that the nonlinear equations of general or irrotational flow can be linearized for small deviations of the dependent variables around the origin. Since the equations are elliptic, a suitable linear transformation of the independent variables will then transform them into the Laplace equation.

For some incident shocks, regular reflection was no longer possible. This corresponded to the wedge angle being too large for a shock to be attached to it. Experimentally, either regular reflection or a three shock configuration

with a slip stream, called Mach reflection, is observed. For weak incident shocks, no solution of the algebraic R-H equations and matching conditions across the slip stream existed so we could not use the techniques of Chapter II which assumed the existence of such solutions. Consequently we examined the equations of viscous flow. The problem was simplified by replacing the mathematical discontinuities, called shocks and slip streams, of non-viscous flow by regions in which viscosity played a role. In other regions the viscosity was neglected.

We first examined the structure of a plane two-dimensional shock region using a linear approximation. We derived the equations when the Mach number of the flow ahead was close to 1. We are particularly interested in this case because it corresponds to those problems where the largest divergence between theory and experiment has been observed. Next we examined a viscous region corresponding to a slip stream in a similar manner,and then applied the equations derived to the problems of two and three shock intersections. The equations obtained admitted the usual non-viscous solutions when the shock curvature was zero. Newton's method in the four variables β, δ, k_R, and k_M, was used to study the equations numerically. The only solution found prescribed a negative shock in violation of the second law of thermodynamics.

If these equations have no physically acceptable solutions we must re-examine the approximations. The first step should be to use quadratic approximations in s for the slip stream and intersection regions since we saw that a linear approximation was inappropriate to the study of boundary layers.

In the solutions of the linear approximations, if they exist, the variables k, the curvatures of the shocks, enter only in the form kw. In Chapter II we discussed the solution of pseudo-stationary flow equations where the variables of length x_1 and x_2 are replaced by the variables $a_1 = x_1/t$ and

$a_2 = x_2/t$ where t is the time. The pseudo-stationary shock curvature $k = \frac{\partial \alpha}{\partial s}$, where s has the dimensions of length/time. Therefore, terms in kw will be functions of a_1, a_2 and t.

This is in contradiction to the experiments of Smith[9] and others, who have observed no measurable deviation from pseudo-stationary behavior in their shock tube experiments. Thus suggests that the region over which viscosity plays a role is too small to be observed in their experiments or that there does exist a form of the solution in which the region affected by viscosity depends on a_1 and a_2 only.

A method of applying the approximations for the shock regions consistently to a boundary value problem is needed before we can come to understand the effect of the bounding walls on the flow patterns for various viscosities. In view of the fact that the equations are different from the non-viscous theory, the surprising result is that there does exist a solution corresponding to the non-viscous theory.

APPENDIX I

The viscous term in the energy equation across a weak shock region is

$$w \frac{\partial}{\partial s} \int_{T_0}^{T} J \, dT = w \frac{\partial}{\partial s} \int_{T_0}^{T_1} \left[v \tan\mu \frac{\partial v}{\partial T} - \frac{v}{\lambda \cos \mu} \frac{\partial v}{\partial s} - v^2 \frac{\partial \xi_2}{\partial T} \right] dT$$

$$= w \Bigg\{ \left[v \right]_{T_0}^{T_1} \left(\frac{\partial v}{\partial s} \tan \mu + v \sec^2 \mu \frac{\partial \mu}{\partial s} \right) + \left[\frac{\partial v}{\partial s} \right]_{T_0}^{T_1} v \tan \mu$$

$$- \frac{\Delta}{\lambda \cos \mu} \left(\frac{\partial v}{\partial s} \right)^2 + \frac{\Delta v}{\lambda \cos \mu} \frac{\partial v}{\partial s} \frac{\partial \lambda}{\partial s} \cos \mu$$

$$- \frac{\Delta v}{(\lambda \cos \mu)^2} \frac{\partial v}{\partial s} \frac{\lambda \partial \mu}{\partial s} \sin \mu - \left[\xi_2 \right]_{T_0}^{T_1} 2v \frac{\partial v}{\partial s}$$

$$- \left[\frac{\partial \xi_2}{\partial s} \right]_{T_0}^{T_1} v^2 \Bigg\}$$

where the terms are evaluated somewhere in (T_0, T_1) unless otherwise indicated. However by our assumptions

$$\mu = O(\epsilon), \quad \frac{\partial v}{\partial s} = O(\epsilon^5), \quad \frac{\partial \mu}{\partial s} = O(\epsilon^4), \quad \frac{\partial \xi_2}{\partial s} = O(\epsilon^6),$$

$$\left[v \right]_{T_0}^{T_1} = O(\epsilon^2), \quad \left[\xi_2 \right]_{T_0}^{T_1} = O(\epsilon^3) \quad \text{and} \quad \Delta = O(\epsilon^{-2}) ;$$

therefore the viscous term has order

$$\epsilon^2 \epsilon^5 \epsilon + \epsilon^2 \epsilon^4 + \epsilon^5 \epsilon + \epsilon^{-2} \epsilon^{10} + \epsilon^{-2} \epsilon^5 \epsilon^4 \epsilon + \epsilon^3 \epsilon^5 + \epsilon^6 = O(\epsilon^6) .$$

A similar proof shows that

$$w \frac{\partial}{\partial s} \int_{T_0}^{T_1} G_i dT = \begin{bmatrix} O(\epsilon^6) \\ \\ O(\epsilon^5) \end{bmatrix}$$

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